Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic, i.e. $f(x + 2\pi) = f(x)$ and even. Its complex Fourier series are given by

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx} , \tag{1}$$

where

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx . \tag{2}$$

Consider the functional

$$\mathcal{J}_f(g) = \int_{[-\pi,\pi]^2} f(x - y) d\mu_x \, d\mu_y , \tag{3}$$

where $d\mu_x = g(x)dx$, $g > 0$ and

$$\int_{-\pi}^{\pi} g(x)dx = 1 . \tag{4}$$

We can now write (3) in the alternative form

$$\mathcal{J}_f(g) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x - y)g(x)g(y)dx\, dy , \tag{5}$$

and set

$$\mathcal{J}_m = \inf_g \mathcal{J}_f(g) .$$

We now claim

Lemma 0.1. *Suppose that the Fourier coefficients of $f(x)$, defined by (2), are all real and non-negative. Let $g_L \equiv 1/(2\pi)$ on $[-\pi, \pi]$. Then,*

$$\mathcal{J}_m = \mathcal{J}_f(g_L) .$$

Furthermore, for any positive g satisfying (4) we have

$$\mathcal{J}_f(g) - \mathcal{J}_f(g_L) = 4\pi^2 \sum_{n\in\mathbb{Z}\setminus\{0\}} a_n |b_n|^2 , \tag{6}$$

in which $\mathbb{Z} \setminus \{0\}$ represents all the integers except for 0, $\{a_n\}_{n=-\infty}^{\infty}$ are the (positive) Fourier coefficients of f, and $\{b_n\}_{n=-\infty}^{\infty}$ are the Fourier coefficients of g.

Proof. We first note that b_n is given by (2) (with g in place of f). Hence, by (4) and (2) we have that

$$b_0 = \frac{1}{2\pi}\,.$$

Let $h = g - g_L$. We first write

$$\mathcal{J}_f(g) - \mathcal{J}_f(g_L) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x-y)h(x)\,dx\,dy + \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x-y)h(y)\,dx\,dy + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x-y)h(x)h(y)dx\,dy\,.$$

Since

$$\int_{-\pi}^{\pi} f(x-y)dx$$

is independent of y, and since

$$\int_{-\pi}^{\pi} h(x)dx = 0\,,$$

we obtain that

$$\mathcal{J}_f(g) - \mathcal{J}_f(g_L) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x-y)h(x)h(y)dx\,dy\,. \tag{7}$$

We next note that

$$f(x-y) = \sum_{n=-\infty}^{\infty} a_n e^{in(x-y)}\,,$$

and, since h is real,

$$h(x) = \sum_{n\in\mathbb{Z}\setminus\{0\}} b_n e^{inx} = \sum_{n\in\mathbb{Z}\setminus\{0\}} \bar{b}_n e^{-inx}\,.$$

Substituting the above into (7) yields

$$\mathcal{J}_f(g) - \mathcal{J}_f(g_L) = \sum_{n_1=-\infty}^{\infty}\sum_{n_2\in\mathbb{Z}\setminus\{0\}}\sum_{n_3\in\mathbb{Z}\setminus\{0\}} a_{n_1} b_{n_2}\bar{b}_{n_3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{in_1(x-y)}e^{-in_2x}e^{in_3y}\,dx\,dy\,.$$

From the orthogonality of Fourier series, i.e.

$$\int_{-\pi}^{\pi} e^{in_1x}e^{-in_2x}\,dx = 2\pi\delta_{n_1n_2}\,,$$

we now get (6). ∎

Consider then the case where

$$g = \frac{1}{N}\sum_{k=0}^{N-1}\delta(x-x_k)\,, \tag{8}$$

where $x_k \in [-\pi,\pi)$. We attempt to minimize $\mathcal{J}_f$ over the set of "functions" (distributions) given by (8) for a given value of N, or equivalently, over all sequences $\{x_k\}_{k=0}^{N-1} \subset [-\pi,\pi)$. We can write $\mathcal{J}_f(g)$ in view of (5) as

$$\mathcal{J}_f(g) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x-y)\frac{1}{N^2}\Big(\sum_{k=0}^{N-1}\delta(x-x_k)\Big)\Big(\sum_{k=0}^{N-1}\delta(y-x_m)\Big)\,dx\,dy =$$

$$\frac{1}{N^2}\sum_{m,k=0}^{N-1}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x-y)\delta(x-x_k)\delta(y-x_m)\,dx\,dy\,,$$

from which we conclude that

$$\mathcal{J}_f(g) = \frac{1}{N^2}\sum_{m,k=0}^{N-1} f(x_m-x_k) = \frac{(N-1)}{N}E + \frac{f(0)}{N}\,, \tag{9}$$

where

$$E = \frac{1}{N(N-1)}\sum_{\substack{m,k=0\\ m\neq k}}^{N-1} f(x_m-x_k)\,.$$

Consequently, the minimizer of $\mathcal{J}_f$ over "functions" defined by (8) is also the minimizer of E.

An immediate conclusion of the above is

Lemma 0.2. *Suppose that the Fourier coefficients of $f(x)$ are all non-negative, and that only a finite number of them are positive. Then, for sufficiently large N we have*

$$\inf_{\{x_k\}_{k=0}^{N-1}\subset[-\pi,\pi)} E = \frac{N}{(N-1)}\Big[\int_{-\pi} f(x)\,dx_1\,dx_2 - \frac{f(0)}{N}\Big]\,. \tag{10}$$

Proof. Set

$$x_k = -\pi + 2\pi \frac{k}{N} \quad 0 \le k \le N-1 \,.$$

It can be easily verified that

$$\bar{b}_q = \frac{(-1)^q}{2\pi} \sum_{k=0}^{N-1} \exp\left\{ i2\pi q \frac{k}{N} \right\}.$$

We then compute the above geometric series

$$\bar{b}_q = \frac{1}{2\pi} \begin{cases} (-1)^N N & \frac{q}{N} \in \mathbb{Z} \\ 0 & \text{otherwise}\,. \end{cases}$$

Hence, by (6)

$$\mathcal{J}_f(g) - \mathcal{J}_f(g_L) = \sum_{j\in\mathbb{Z}\setminus\{0\}} a_{jN} \,. \tag{11}$$

Suppose now that there exists some $M_0 \in \mathbb{N}$ such that for every $|n| > M_0$ we have $a_n = 0$. It follows by (11) that $\mathcal{J}_f(g) = \mathcal{J}_f(g_L)$. The lemma easily follows by (9). ∎

We now state and prove some well known result from Fourier analysis

Lemma 0.3. *Let f be 2π-periodic with k continuous derivatives ($f \in C^k(\mathbb{R})$). Let $\{a_n\}_{n=1}^{\infty}$ denote its Fourier coefficients, given by (2). Then, there exists $C_k > 0$, independent of n, such that*

$$|a_n| \le \frac{C_k}{n^k} \,. \tag{12}$$

Proof. Integration by parts yields, by (2), that

$$2\pi a_n = \frac{f(x)}{in} e^{-inx} \Big|_0^{2\pi} + \frac{1}{in} \int_0^{2\pi} f'(x) e^{-inx}\, dx \,.$$

As $f(0) = f(2\pi)$ we obtain that

$$2\pi a_n = \frac{1}{in} \int_0^{2\pi} f'(x) e^{-inx}\, dx \,.$$

We can now continue to integrate by parts $k-1$ more times to obtain that

$$2\pi a_n = \frac{1}{(in)^k} \int_0^{2\pi} f^{(k)}(x) e^{-inx}\, dx\,. \tag{13}$$

Since $f \in C^k(\mathbb{R})$ we have that

$$\sup_{x\in[0,2\pi]} |f^{(k)}(x)| = C_k < \infty\,.$$

Consequently, by the triangle (Minkowski) inequality we obtain that

$$\Big| \int_0^{2\pi} f^{(k)}(x) e^{-inx}\, dx \Big| \le \int_0^{2\pi} |f^{(k)}(x)|\,|e^{-inx}|\, dx \le 2\pi C_k\,.$$

Substituting into (13) yields (12). ∎

We say that $f \in C^\infty(\mathbb{R})$ if $f \in C^k(\mathbb{R})$ for all $k \ge 1$.

Corollary 0.4. *Let $f \in C^\infty(\mathbb{R})$ and let $\{a_n\}_{n=1}^\infty$ denote its Fourier coefficients. Then, for every $k \ge 1$ there exists $C_k > 0$ such that*

$$|a_n| \le \frac{C_k}{n^k}\,.$$

The converse statement is also true.

Lemma 0.5. *Let $\{a_n\}_{n=-\infty}^\infty \subset \mathbb{C}$. Suppose that for some $k \ge 1$ we have for some $C > 0$, that*

$$|a_n| \le \frac{C}{n^k}\,.$$

Let

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}\,. \tag{14}$$

Then $f \in C^{k-1}(\mathbb{R})$.

Without proof.

We say that f is analytic at $z_0 \in \mathbb{C}$ if the series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n \tag{15}$$

converges in some neighborhood of z_0. The Fourier coefficient of an analytic function have the following property

Lemma 0.6. *Let f be analytic in the strip $|\Im z| < \alpha$ ($\Im z$ is the imaginary part of z) for some $\alpha > 0$. Let $\{a_n\}_{n=1}^{\infty}$ denote its Fourier coefficients. Then, for every $\beta < \alpha$ there exists $C_\beta > 0$ such that*

$$|a_n| \leq C_\beta e^{-\beta|n|} \,. \tag{16}$$

Conversely, if (16) holds for some $\beta > 0$ and f is defined by (14), then f is analytic in the strip $|\Im z| < \beta$

Without proof once again. An immediate corollary follows.

Corollary 0.7. *Let $f \in C^k(\mathbb{R})$ for some $k \geq 2$. Then for some $C > 0$,*

$$0 \leq \mathcal{J}_f(g) - \mathcal{J}_f(g_L) \leq \frac{C}{N^k} \,. \tag{17}$$

Furthermore, if f is analytic in the strip $|\Im z| < \alpha$ then

$$0 \leq \mathcal{J}_f(g) - \mathcal{J}_f(g_L) \leq C_\beta e^{-\beta N} \,, \tag{18}$$

for all $\beta < \alpha$.

Proof. Follows immediately from (11) and either (12) or (16). ∎

Conjecture 0.8. *If f has positive Fourier coefficients, and is sufficiently smooth, then for sufficiently large N the minimum of $\mathcal{J}_f(g)$ over (8) is obtained for*

$$x_k = \frac{k}{N} \quad 0 \leq k \leq N-1 \,.$$

The minimum is unique up to translation.

The goal is to check the conjecture numerically, and see what happens if some of the coefficients are negative.