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המכללה האקדמית להנדסה
ORT Braude College

Department
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המחלקה
למתמטיקה

פרויקט מסכם לתואר בוגר במדעים
במתמטיקה שימושית (B.Sc)

Final Project in the Applied Mathematics Bachelor's Degree (B.Sc)

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Asymptotic behavior of parabolic-type semigroups of holomorphic mappings

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Contents

1 Introduction

1.1 Dynamical systems

The study of dynamical systems, as its name would suggest, deals with the theoretical and practical analysis of the evolution of processes. It describes the motion of a process, attempts to predict its future state as well as understand the limitations of these predictions. Some of these processes are mathematical descriptions of various natural phenomena, and thus those who study dynamical systems find themselves touching on a wide range of subjects: from the computation of sea currents and waves to the statistics of digits in the decimal expansion of π; from the assessment of the damage from a tsunami to the prediction of the behavior of a roulette ball. All these systems share the property of changing over time. These changes are governed by the specific laws or dynamics relevant to each system - Newtonian mechanics, the laws of probability, linear transformations and so on.

Each dynamical system has two main parts that define it, namely a phase space and a function describing its dynamics. The phase space is a set containing all the possible states of the system at hand (with each state being all the information about the system at a given time). Complementarily, the dynamic function is a mapping between these states. Although this definition suggests a very deterministic system, where the application of the dynamic function to a selected initial state gives rise to a pre-determined future state, there are systems in which this rule is not necessarily true. Some dynamic systems are used to predict chaotic or random processes, and some are used to analyze the current state of a given system and try to deduce earlier states it may possibly have originated form.

The choice of dynamic function plays a great role in the accuracy of the model in being studied. Most times the exact function describing a process is unknown, and finding an approximation is a challenging task. Moreover, functions need not be static, and are generally time-dependent and defined as $\Phi = \Phi(t, \omega) \in T \times \Omega \to \Omega$ (meaning they are a function of both time and state). This flexibility allows a model to correctly describe a changing environment or decreasing information about a process as time passes.

In discussing dynamical systems, normed, complete spaces (Banach spaces) are usually chosen to describe the phase space, because it is quite helpful if the possibility exists to determine whether or not two states are "close", or at least "closer" than others. In this way it is possible to tell where a process is heading and investigate its asymptotic behavior, as well as learn about its stability when changing initial values.

A mathematical definition of a dynamical system is a triple (T, Ω, Φ), defining the time-space, phase-space and dynamic function. It is sometimes useful to observe the function for all points in the set at a given time (notated as $\Phi_{t_0}(\omega)$), or as a trajectory through time from a given point ($\gamma_{\omega_0}(t)$), and we will expand on these ideas later on.

1.2 Semigroups

A semigroup (G; $*$) is generally defined as a set G on which the binary operator $*$ is defined, and the associativity law $(x*y)*z = x*(y*z)$ holds (and that is the only requirement!). In the case of dynamical systems, the composition of functions operator (notated $\circ$) is usually chosen as the semigroup operator, and the sets are usually functions mapping from the phase space onto itself. Thus, they are known as *composition semigroups.* This area of study, namely composition semigroups on spaces of analytic functions will be considered in this project. For these semigroups we will assume that the set G is closed under the composition operator ($f_1, f_2 \in F \rightarrow f_1 \circ f_2 \in F$), and that a unity exists, i.e. $\exists f_u \in F : \forall f \in F, f_u \circ f = f \circ f_u = f$.

The choice of a set representing time is also of importance. This set, often called a monoid and defined as a semigroup with an identity element, is a basic feature of any dynamical system. It governs the space through which the process evolves, and can usually be thought of as "time", although geometrical interpretations are possible as well. The types of systems have different dynamic functions or a different monoid describing time. Time can, for example, be a negative number for those systems involved in deduction of past events, or a real number for systems describing continuous processes. In fact, the choice of monoid used in defining a system is a useful way of classifying it, distinguishing between maps (using discrete, positive time $\mathbb{N}$), invertible maps ($\mathbb{Z}$), semi-flows ($\mathbb{R}^+$) and flows ($\mathbb{R}$), for example.

In this project, our chosen monoid will be that of positive, continuous time ($\mathbb{R}^+$), as we are investigating the asymptotic future behavior only. The functions preoccupying us will be holomorphic (analytic) maps over the sets of the unit disk or the right half of the complex plane.

2 Preliminaries

We denote by $\mathrm{Hol}(D, \mathbb{C})$ the set of all holomorphic functions on a domain $D \subset \mathbb{C}$, and by $\mathrm{Hol}(D)$ the set of all holomorphic self-mappings of D.

We say that a family $S = \{F_t\}_{t \geq 0} \subset \mathrm{Hol}(D)$ is *a one-parameter continuous semigroup on* D (semigroup, in short) if:

(i) $F_t(F_s(z)) = F_{t+s}(z)$ for all $t, s \geq 0$ and $z \in D$[1], and

(ii) $\lim\limits_{t \to 0^+} F_t(z) = z$ for all $z \in D$.

In the case when D is the open unit disk $\Delta = \{z : |z| < 1\}$, it follows from a result of E. Berkson and H. Porta [1] that each semigroup is differentiable with respect to $t \in \mathbb{R}^+ = [0, \infty)$. So, for each one-parameter continuous semigroup $S = \{F_t\}_{t \geq 0} \subset \mathrm{Hol}(\Delta)$, the limit

$$\lim_{t \to 0^+} \frac{z - F_t(z)}{t} = f(z), \quad z \in \Delta,$$

exists and defines a holomorphic mapping $f \in \mathrm{Hol}(\Delta, \mathbb{C})$. This mapping f is called the *(infinitesimal) generator of* $S = \{F_t\}_{t \geq 0}$. Moreover, the function $u(t, z) := F_t(z)$, $(t, z) \in \mathbb{R}^+ \times \Delta$, is the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial u(t,z)}{\partial t} + f(u(t,z)) = 0, \\ u(0, z) = z, \quad z \in \Delta. \end{cases} \tag{2.1}$$

In the same paper, Berkson and Porta proved that a function $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ is a semigroup generator if and only if there exist both some function $p \in \mathrm{Hol}(\Delta, \mathbb{C})$ with $\mathrm{Re}\, p(z) \geq 0$, and a point $\tau \in \overline{\Delta}$, such that

$$f(z) = (z - \tau)(1 - z\bar{\tau})p(z). \tag{2.2}$$

This representation is unique. Moreover, if S contains neither the identity mapping nor an elliptic automorphism of Δ, then τ is a unique attractive fixed point of S, i.e., $\lim\limits_{t \to \infty} F_t(z) = \tau, \forall\, z \in \Delta$, and $\lim\limits_{r \to 1^-} F_t(r\tau) = \tau$. The point τ is called the *Denjoy–Wolff point of* S.

Recently the asymptotic behavior of semigroups including the local geometry of semigroup trajectories near their boundary Denjoy–Wolff

[1] This is sometimes called the "semigroup property". Note that it means the semigroup is closed under addition over the monoid $T = \mathbb{R}^+$.

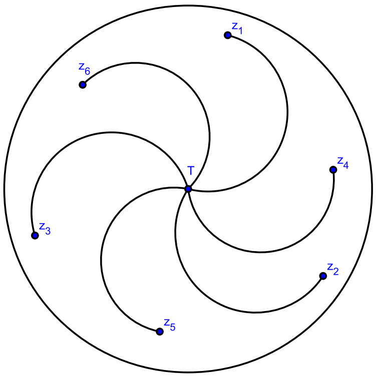

Figure 2.1: The interior Denjoy–Wolff point is the limit of all trajectories regardless of starting point.

point have been attracting a lot of attention. It was shown in [4] that if $\tau \in \partial\Delta$, then the angular derivative $f'(\tau) = \angle \lim\limits_{z\to\tau} f'(z) = \angle \lim\limits_{z\to\tau} \dfrac{f(z)}{z-\tau}$ of f at the point $\tau \in \partial\Delta$ exists and is a non-negative real number. One distinguishes between two cases: (a) $f'(\tau) > 0$ (the *hyperbolic case*), and (b) $f'(\tau) = 0$ (the *parabolic case*). We will limit our discussion in this project to semigroups of the parabolic type, but to underline the essential difference between the two types we will bring to the reader's attention the following known fact: there is an essential difference between semigroups of the hyperbolic and parabolic types; in the hyperbolic case, the limit tangent line depends on the initial point of the trajectory, whereas in the parabolic case all the trajectories have the same tangent line (if such a line exists). See Fig. 2.2 and [2, 8, 10, 5, 9] for details. In the hyperbolic case the semigroup trajectories always have tangent lines passing through the Denjoy–Wolff point, in the parabolic case this claim is not certain.

Since any trajectory $\gamma_z = \{F_t(z),\ t \geq 0\}\,,\ z \in \Delta$, is an analytic curve, it has a finite curvature at each its point $F_s(z)$. Following the notion in [6], for each $z \in \Delta$ we denote by $\kappa(z,s)$ the curvature of the trajectory γ_z at the point $F_s(z)$ and by $\kappa(z)$ the *limit curvature of the trajectory*: $\kappa(z) := \lim\limits_{s\to\infty} \kappa(z,s)$ (if it exists). Therefore, the above question is equivalent to the following one: *When does the limit curvature of a semigroup trajectory exist finitely?*

From now on, we assume without loss of generality that $\tau = 1$. It turns out (see [8]) that two hyperbolic type semigroups having similar

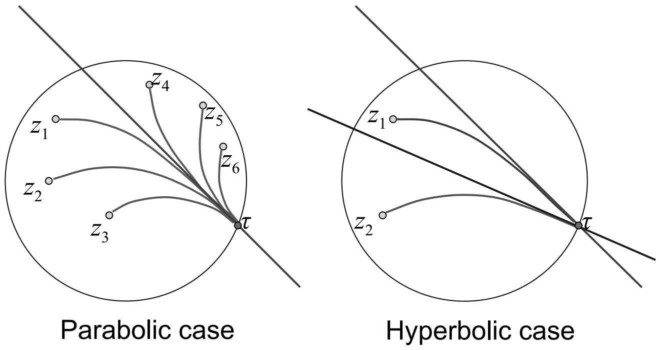

Figure 2.2: The existence and uniqueness (for different initial points) of the tangent line depends on the semigroup type.

asymptotic behavior actually coincide up to re-scaling. This fact is not longer true for parabolic type semigroups. Moreover, if the generator f of a semigroup $S = \{F_t\}_{t\geq 0}$ admits the representation

$$f(z) = b(1-z)^2 + o((1-z)^2),$$

then for each $z \in \Delta$, the limit tangent line to the trajectory $\gamma_z = \{F_t(z)\}_{t\geq 0}$ exists with

$$\lim_{t\to\infty} \arg(1 - F_t(z)) = -\arg b,$$

hence, does not depend on $z \in \Delta$ as well as on the remainder $o((1-z)^2)$ (see [8] and Theorem 2.2 (i) below).

To answer the questions above for a semigroup $S = \{F_t\}_{t\geq 0}$ generated by $f \in \mathrm{Hol}(\Delta, \mathbb{C})$, we apply a linearization model given by Abel's functional equation

$$h\left(F_t(z)\right) = h(z) + t. \tag{2.3}$$

It is rather simple to see that the function $h : \Delta \mapsto \mathbb{C}$ defined by

$$h'(z) = -\frac{1}{f(z)}, \quad h(0) = 0, \tag{2.4}$$

solves equation (2.3). This function is univalent and convex in the positive direction of the real axis due to (2.3). Sometimes it is called the *Kœnigs function* for the semigroup (see [2, 8, 10, 14] and [9]).
In the parabolic case we can see more subtlety than in its hyperbolic counterpart, in that there are some semigroups which converge to the boundary Denjoy–Wolff point tangentially, as well as known examples of non-tangentially converging semigroups (see [2, 3, 5, 8]).

M. D. Contreras and S. Díaz-Madrigal in [2] have considered the set $\mathrm{Slope}^+(\gamma_z)$ of all accumulation points (as $t \to \infty$) of the function $\arg(1 - F_t(z))$ and proved that these sets coincide for all $z \in \Delta$. In addition, they have proven that if for a function h defined by (2.4), the image $h(\Delta)$ lies in a horizontal half-plane, then all the trajectories γ_z tend tangentially to $\tau = 1$. In addition, $\mathrm{Slope}^+(\gamma_z)$ is a single point which is equal to either $\pi/2$ or $-\pi/2$. In general the question whether $\mathrm{Slope}^+(\gamma_z)$ is a singleton is still open (see [2, 10]).

Inasmuch as we are updated, all known results in this vein require some smoothness conditions at the Denjoy–Wolff point. For example, if the semigroup generator f is twice differentiable at the boundary Denjoy–Wolff point $\tau = 1$, then all the trajectories γ_z converge to this point tangentially if and only if $\mathrm{Re}\, f''(1) = 0$ (see [8]).

An advanced question in this study, recently raised by Elin and Shoikhet [6] is: how close is a semigroup trajectory to its tangent line? In particular, one can ask: *Is there such a circle sharing the same tangent line at the Denjoy-Wolff point, so that the trajectories lie between this circle and the line?* See Fig. 2.3.

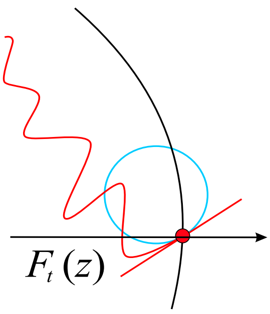

Figure 2.3: An advanced question: *Is there such a circle sharing the same tangent line at the Denjoy-Wolff point, so that the trajectories lie between this circle and the line?*

Furthermore, it may happen that for each $z \in \Delta$, there is a horodisk $D(\tau, k) := \{\zeta \in \Delta : \ d(\zeta, \tau) < k\}$, $k = k(z)$, internally tangent to the unit circle at the point τ, such that the trajectory γ_z lies outside $D(\tau, k)$. In this case we say that the semigroup S converges to τ *strongly tangentially*. It is clear that the supremum of the radii of such horodisks coincides with the limit curvature radius. Conversely, if a semigroup converges tangentially but not strongly tangentially, then its trajectories have infinite limit curvature.

For generators differentiable three times at the boundary Denjoy–Wolff point τ with $f'''(\tau) = 0$, the following rigidity phenomenon was established in [13]: *The semigroup S generated by f converges to τ strongly tangentially if and only if it consists of parabolic automorphisms of Δ (i.e., its trajectories have finite curvature).*

The question on the finiteness of the limit curvature is a general problem which is also relevant for non-tangentially converging semigroups. In addition, one can ask: might $\kappa(z)$ be finite for some points $z \in \Delta$ and infinite for others? In [6], the following answers were presented for semigroups generated by functions which are $(3+\varepsilon)$-smooth at the Denjoy–Wolff point.

Theorem 2.1 *Let $S = \{F_t\}_{t\geq 0}$ be a semigroup generated by $f \in \mathbb{C}^{3+\varepsilon}(1)$, i.e.,*

$$f(z) = b(1-z)^2 + c(1-z)^3 + R(z), \tag{2.5}$$

where $R \in \operatorname{Hol}(\Delta, \mathbb{C})$, $\displaystyle\lim_{z\to 1} \frac{R(z)}{(1-z)^{3+\varepsilon}} = 0$, and let $b \neq 0$.

(a) If $\operatorname{Im} \dfrac{c}{b^2} \neq 0$, then all of the trajectories have infinite limit curvature, i.e., $\kappa(z_0) = \infty$ for each $z_0 \in \Delta$.

(b) Otherwise, if $\operatorname{Im} \dfrac{c}{b^2} = 0$, then each trajectory $\{F_t(z_0),\ t \geq 0\}$ has a finite limit curvature, namely, $\kappa(z_0) = \left|\frac{2C}{b}\right|$, where

$$C = |b|^2 \operatorname{Im} h(z_0) + \operatorname{Im} b + \int_0^\infty \operatorname{Im} \left(\frac{f\left(F_s(1)\right) \overline{b}}{(1 - F_s(0))^2} - \frac{c\overline{b}}{b(s+1)} \right) ds.$$

Thus, under the above assumptions if $\kappa(z)$ is finite for some $z \in \Delta$, then it must be finite for all $z \in \Delta$.

Once again, one can see that there is a cardinal difference between semigroups of hyperbolic and parabolic types: in the hyperbolic case with some smoothness conditions the limit curvature is always finite, while in the parabolic case the limit curvature may be infinite. At the same time, it follows from [13] that if the second derivative $f''(1)$ is purely imaginary, then the third derivative $f'''(1)$ should be real. Thus, an immediate consequence of part (b) of Theorem 2.1 is the following fact:

Corollary 2.1 *Let $\{F_t\}_{t\geq 0}$ be a semigroup of holomorphic self-mappings of the open unit disk Δ generated by $f \in \mathbb{C}^{3+\varepsilon}(1)$ of the form (2.5) with $b \neq 0$. If $\operatorname{Re} b = 0$, then each semigroup trajectory converges to $\tau = 1$ strongly tangentially.*

As a matter of fact, theorem 2.1 is based on the following general result which contains complete quantitative characteristics of the asymptotic behavior for semigroups generated by functions smooth enough at the boundary Denjoy–Wolff points.

Theorem 2.2 (see [6] and [8]) *Let $S = \{F_t\}_{t\geq 0}$ be a continuous semigroup of holomorphic self-mappings of the open unit disk Δ, and let f be its generator.*

(i) Suppose that f admits the following representation:

$$f(z) = b(1-z)^2 + R(z), \tag{2.6}$$

where $R \in \operatorname{Hol}(\Delta, \mathbb{C})$, $\lim\limits_{z\to 1} \dfrac{R(z)}{(1-z)^2} = 0$. *Then*

$$\frac{1}{1-F_t(z)} = -bt + G(z,t), \quad \textit{where } \lim_{t\to\infty} \frac{G(z,t)}{t} = 0, \tag{2.7}$$

and

$$\lim_{t\to\infty} \left(\frac{1}{1-F_t(z)} - \frac{1}{1-F_t(0)} \right) = -bh(z). \tag{2.8}$$

(ii) If $b \neq 0$ and R in (3.12) is of the form $R(z) = c(1-z)^3 + R_1(z)$ with $R_1 \in \operatorname{Hol}(\Delta, \mathbb{C})$, $\lim\limits_{z\to 1} \dfrac{R_1(z)}{(1-z)^3} = 0$, *i.e., f admits the representation:*

$$f(z) = b(1-z)^2 + c(1-z)^3 + R_1(z), \tag{2.9}$$

then

$$\frac{1}{1-F_t(z)} = -bt - \frac{c}{b}\log(t+1) + G_1(z,t), \tag{2.10}$$

where $\lim\limits_{t\to\infty} \dfrac{G_1(z,t)}{\log(t+1)} = 0$, *and*

$$\lim_{t\to\infty} t \left(\frac{1}{1-F_t(z)} - \frac{1}{1-F_t(0)} + bh(z) \right) = -\frac{c}{b}\, h(z). \tag{2.11}$$

3 Main Results

In the present project we are going to study similar characteristics for asymptotic behavior of the parabolic type semigroups generated by $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ of the form:

$$f(z) = a(1-z)^{1+r} + R(z), \tag{3.12}$$

where $R \in \mathrm{Hol}(\Delta, \mathbb{C})$, $\lim\limits_{z\to 1} \dfrac{R(z)}{(1-z)^{1+r}} = 0$.
This case is a generalized version of theorem 2.1, valid for any $r \in (0, 2]$. The conclusions derived in [6] are a specific case of this new, generalized formula, which can be applied to a broader set of infinitesimal generator functions. Clearly, setting $r = 1$ will re-produce those earlier results. Although the first part of this result was already described in [10], the second part is new. In sections 4 and 5 we provide new proofs for both old and new assertions, for the sake of completeness. There, given a little more information about the generator's form, we intend to show that the following result can be obtained:

Theorem 3.1 *Let $S = \{F_t\}_{t\geq 0}$ be a continuous semigroup of holomorphic self-mappings of the open unit disk Δ and let f be its generator.*

(i) Suppose that f admits the representation (3.12).
Denote $\gamma = 2^{r-1}ar$, then

$$\frac{2}{1-F_t(z)} = (\gamma t)^{\frac{1}{r}} + G(z,t), \quad \textit{where } \lim_{t\to\infty} \frac{G(z,t)}{t^{\frac{1}{r}}} = 0, \tag{3.13}$$

and

$$\lim_{t\to\infty} \left(\left(\frac{1+F_t(z)}{1-F_t(z)} \right)^r - \left(\frac{1+F_t(0)}{1-F_t(0)} \right)^r \right) = \gamma h(z). \tag{3.14}$$

(ii) If $a \neq 0$ and R in (3.12) is of the form $R(z) = b(1-z)^{1+2r} + R_1(z)$ with $R_1 \in \mathrm{Hol}(\Delta, \mathbb{C})$, $\lim\limits_{z\to 1} \dfrac{R_1(z)}{(1-z)^{1+2r}} = 0$, i.e., f admits the representation:

$$f(z) = a(1-z)^{1+r} + b(1-z)^{1+2r} + R_1(z). \tag{3.15}$$

Denote $\lambda = \frac{2^r b}{a}$. *Then*

$$\left(\frac{2}{1-F_t(z)}\right)^r = \gamma t + \lambda \log(t+1) + G_1(z,t), \tag{3.16}$$

where $\lim\limits_{t\to\infty} \dfrac{G_1(z,t)}{\log(t+1)} = 0$. *Moreover,*

$$\lim_{t\to\infty} (t+1)\left(\frac{2}{1-F_t(z)} - \frac{2}{1-F_t(0)} - \gamma h(z)\right) = \lambda h(z), \tag{3.17}$$

(3.16) can be re-written in a more pleasing way. We will show this more optimized formula here and add its derivation in section 5.

$$\frac{2}{1-F_t(z)} = (\gamma t)^{\frac{1}{r}} + \frac{\lambda}{r}(\gamma t)^{\frac{1}{r}-1}\log(t+1) + \Gamma_1(t), \tag{3.18}$$

with $\lim\limits_{t\to\infty} \dfrac{\Gamma_1(t)}{\left(\dfrac{\log(t+1)}{t}\right)} = 0.$

4 The right half-plane model

For ease of proof we first transfer the study of the semigroup behavior from the open unit disk to right half-plane by using the Cayley transform

$$C(z) = \frac{1+z}{1-z}.$$

Now, given a semigroup $S = \{F_t\}_{t\geq 0} \subset \mathrm{Hol}(\Delta)$ with the Denjoy–Wolff point $\tau = 1$, we construct the semigroup $\Sigma = \{\Phi_t\}_{t\geq 0}$ of holomorphic self-mappings of the right half-plane $\Pi = \{w \in \mathbb{C} : \ \mathrm{Re}\, w > 0\}$ as follows:

$$\Phi_t(w) = C \circ F_t \circ C^{-1}(w) \tag{4.1}$$

with the Denjoy–Wolff point at ∞

$$\Phi_t(w) = C \circ F_t \circ C^{-1}(\infty) = C \circ F_t(\tau) = C(\tau) = \infty.$$

If S is continuous (hence, differentiable) in t, then Σ is too. More precisely, let f be the infinitesimal generator of S. Then by (2.2), f must be of the form $f(z) = -(1-z)^2 p(z)$ with $\mathrm{Re}\, p(z) \geq 0, \ z \in \Delta$.

Differentiating Φ_t given by (4.1) at $t = 0^+$, we conclude that Σ is generated by the mapping $-\phi$,

$$\phi(w) = -C'\left(F_t\left(C^{-1}(w)\right)\right)\left(F_t'\left(C^{-1}(w)\right)\right) = 2p\left(C^{-1}(w)\right) \tag{4.2}$$

(cf., [7, Lemma 3.7.1]). So, $\phi \in \mathrm{Hol}(\Pi, \overline{\Pi})$ and the semigroup $\Sigma = \{\Phi_t\}_{t\geq 0}$ satisfies the Cauchy problem

$$\begin{cases} \dfrac{\partial \Phi_t(w)}{\partial t} = \phi\left(\Phi_t(w)\right), \\ \Phi_t(w)\big|_{t=0} = w, \quad w \in \Pi. \end{cases} \tag{4.3}$$

Concerning the Kœnigs function h defined by (2.4), one can modify it to $\sigma := h \circ C^{-1}$. By direct calculations we check that for all $w \in \Pi$ this modified function satisfies Abel's functional equation

$$\sigma\left(\Phi_t(w)\right) = h \circ \left(F_t \circ C^{-1}(w)\right) = h \circ \left(C^{-1}(w)\right) + t = \sigma(w) + t, \tag{4.4}$$

as well as the initial value problem:

$$\sigma'(w) = \frac{1}{\phi(w)}, \quad \sigma(1) = 0. \tag{4.5}$$

It was already mentioned that the angular derivative $a = f'(1)$ always exists and $a \geq 0$. If, in addition, $f \in \mathbb{C}^{1+2r}(1)$, then f admits representation

$$f(z) = a(1-z)^{1+r} + b(1-z)^{1+2r} + R(z) \quad z \in \Delta, \tag{4.6}$$

with

$$\lim_{z\to 1^-} \frac{R(z)}{(1-z)^{1+2r}} = 0.$$

Suppose now that the semigroup S generated by f is of parabolic type. For this type, as proven in [10], we have $0 < r \leq 2$ in (4.6).

By the Berkson–Porta representation (2.2) with $\tau = 1$ and formulas (4.2), (4.6), the function ϕ can be represented as follows:

$$\phi(w) = -2^{r-1}a(w+1)^{1-r} - 2^{2r-1}b(w+1)^{1-2r} + \rho(w), \tag{4.7}$$

where

$$\angle \lim_{w\to\infty} \frac{\rho(w)}{(w+1)^{1-2r}} = 0 \tag{4.8}$$

Since Σ has the Denjoy–Wolff point at ∞, we have by Julia's Lemma (see, for example, [11, 12, 9]) that $\operatorname{Re}\Phi_t(w)$ is an increasing function in $t \geq 0$. The tangential convergence of the semigroup means that the function $\dfrac{\operatorname{Im}\Phi_t(w)}{\operatorname{Re}\Phi_t(w)}$ is unbounded as t tends to infinity. Roughly speaking, the semigroup converges tangentially when $|\operatorname{Im}\Phi_t(w)|$ grows faster than $\operatorname{Re}\Phi_t(w)$. Moreover, the original semigroup $S = \{F_t\}_{t\geq 0}$ converges strongly tangentially if and only if the function $\operatorname{Re}\Phi_t(w)$ is bounded for each w with $\operatorname{Re} w > 0$. For this reason, strongly tangentially convergent semigroups were referred to in [3] as semigroups of *finite shift*; and weakly tangentially convergent semigroups as semigroups of *infinite shift*. Therefore, a semigroup S converges strongly tangentially if and only if each trajectory of the semigroup Σ defined by (4.1) has a vertical asymptote. More generally, *a semigroup trajectory in the open unit disk has a finite limit curvature if and only if the corresponding trajectory in the right half-plane has an asymptote as $t \to \infty$.*

Next we will consider parabolic type semigroups in the right half-plane.

5 Proofs

First let us consider the case when parabolic type semigroup $\Sigma = \{\Phi_t\}_{t\geq 0}$ of the right half-plane generated by $-\phi$ of form

$$\phi(w) = A(w+1)^{1-r} + \varrho(w), \tag{5.1}$$

where $\varrho \in \operatorname{Hol}(\Pi, \mathbb{C})$, $\quad \angle \lim\limits_{w\to\infty} \dfrac{\varrho(w)}{(w+1)^{1-r}} = 0.$

Theorem 5.1 *Let $\{\Phi_t\}_{t\geq 0} \in \operatorname{Hol}(\Pi)$ be a semigroup of parabolic type with the Denjoy–Wolff point at ∞ generated by mapping $-\phi$.*

Then

$$\Phi_t(w) = (\gamma t)^{\frac{1}{r}} + \Gamma(w,t), \quad \textit{where } \lim_{t\to\infty} \frac{\Gamma(w,t)}{t^{\frac{1}{r}}} = 0, \tag{5.2}$$

and

$$\lim_{t\to\infty} \left((\Phi_t(w))^r - (\Phi_t(1))^r \right) = \gamma\sigma(w), \tag{5.3}$$

where σ is defined by (4.5) and $\gamma = Ar$.

Proof. Fix $w \in \Pi$ and consider $\Phi_t(w)$ as a (complex valued) function of the real variable t. This function tends to infinity as $t \to \infty$. Thus, by the L'Hospital rule

$$\lim_{t\to\infty} \frac{(\Phi_t(w)+1)^r}{t+1} = \lim_{t\to\infty} \frac{r\,(\Phi_t(w)+1)^{r-1}\,\phi\,(\Phi_t(w))}{1} =$$

$$\lim_{t\to\infty} r\left(A + \frac{\varrho\,(\Phi_t(w))}{(\Phi_t(w))^{1-r}}\right) = \gamma. \tag{5.4}$$

Thus,

$$\lim_{t\to\infty} \frac{\Phi_t(w)}{t^{\frac{1}{r}}} = \lim_{t\to\infty} \frac{\Phi_t(w)}{t^{\frac{1}{r}}} \cdot \frac{\Phi_t(w)+1}{\Phi_t(w)} \cdot \left(\frac{t}{t+1}\right)^{\frac{1}{r}} = \lim_{t\to\infty} \frac{(\Phi_t(w)+1)}{(t+1)^{\frac{1}{r}}} = \gamma^{\frac{1}{r}}.$$

This proves formula (5.2). Furthermore,

$$\lim_{t\to\infty} \left((\Phi_t(w))^r - (\Phi_t(1))^r\right)$$

$$= \lim_{t\to\infty} \int_1^w \left((\Phi_t(z))^r\right)' dz = \lim_{t\to\infty} r \int_1^w (\Phi_t(z))^{r-1}\, \frac{\phi\,(\Phi_t(z))}{\phi(z)} dz$$

$$= \lim_{t\to\infty} r \int_1^w \left(\frac{\Phi_t(z)}{\Phi_t(z)+1}\right)^{r-1} \frac{1}{\phi(z)} \left(A + \frac{\varrho\,(\Phi_t(z))}{(\Phi_t(z))^{1-r}}\right) dz$$

$$= \gamma \int_1^w \frac{dz}{\phi(z)} = \gamma\sigma(w)$$

by (5.4). ■

It turns out that in case the function ρ can be represented in the form $\rho(w) = B(w+1)^{1-2r} + \varrho_1(w)$ such that $\lim\limits_{w\to\infty} \dfrac{\varrho_1(w)}{(w+1)^{1-2r}} = 0$, we can achieve a more precise estimate for the asymptotic behavior of $\Sigma = \{\Phi_t\}_{t\geq 0}$. This is, in fact, the main part of this project, transformed into the half-plane. Using the inverse Cayley's transform we will convert its proof given here to show the validity of the correlating assertions in Δ, made in theorem 3.1.

Theorem 5.2 *Let $\{\Phi_t\}_{t\geq 0} \in \mathrm{Hol}(\Pi)$ be a semigroup of parabolic type with the Denjoy–Wolff point at ∞ generated by mapping $-\phi$.*

$$\phi(w) = A(w+1)^{1-r} + B(w+1)^{1-2r} + \varrho_1(w), \tag{5.5}$$

with $A \neq 0$, $\varrho_1 \in \mathrm{Hol}(\Pi, \mathbb{C})$ *and* $\lim_{w\to\infty} \dfrac{\varrho_1(w)}{(w+1)^{1-2r}} = 0.$

Then, denoting $\gamma = Ar$ *and* $\lambda = \frac{B}{A}$, *we have*

$$\Phi_t(w) = (\gamma t)^{\frac{1}{r}} - 1 + \frac{\lambda}{r}(\gamma t)^{\frac{1-r}{r}} \log(t+1) + \Gamma(w,t) \tag{5.6}$$

with $\lim_{t\to\infty} \dfrac{\Gamma(w,t)}{t^{\frac{1-r}{r}} \log(t+1)} = 0.$[2] *Moreover,*

$$\lim_{t\to\infty} (t+1)\left((\Phi_t(w)+1)^r - (\Phi_t(1)+1)^r - \gamma\sigma(w)\right) = \lambda\sigma(w). \tag{5.7}$$

Proof. First we show that

$$\lim_{t\to\infty} \frac{1}{\log(t+1)} \left((\Phi_t(w)+1)^r - \gamma t - \lambda \log(t+1)\right) = 0. \tag{5.8}$$

One can calculate

$$\frac{d}{ds}\left((\Phi_s(w)+1)^r - \gamma s - \lambda\log(s+1)\right) =$$

$$= r\left(\Phi_s(w)+1\right)^{r-1} \phi\left(\Phi_s(w)\right) - \gamma - \lambda\frac{1}{s+1} =$$

$$= \frac{r}{s+1}\left(B\frac{s+1}{(\Phi_s(w)+1)^r} - \frac{\lambda}{r} + \frac{\rho_1(\Phi_s(w))(s+1)}{(\Phi_s(w)+1)^{1-r}}\right)$$

$$= \frac{r}{s+1}\left(B\left(\frac{s+1}{(\Phi_s(w)+1)^r} - \frac{1}{\gamma}\right) + \frac{\rho_1(\Phi_s(w))}{(\Phi_s(w)+1)^{1-2r}}\frac{s+1}{(\Phi_s(w)+1)^r}\right),$$

where by (5.4) and (5.5) it follows that

$$\lim_{s\to\infty}\left(B\left(\frac{s+1}{(\Phi_s(w)+1)^r} - \frac{1}{\gamma}\right) + \frac{\rho_1(\Phi_s(w))}{(\Phi_s(w)+1)^{1-2r}}\frac{s+1}{(\Phi_s(w)+1)^r}\right) = 0.$$

Therefore, it follows that for each $\varepsilon > 0$ there exists $t_0 > 1$ such that for all $s > t_0$ we have

$$\left|B\left(\frac{s+1}{(\Phi_s(w)+1)^r} - \frac{1}{\gamma}\right) + \frac{\rho_1(\Phi_s(w))}{(\Phi_s(w)+1)^{1-2r}}\frac{s+1}{(\Phi_s(w)+1)^r}\right| < \frac{\varepsilon}{r}.$$

[2] if $r < 1$ we can insert the constant 1 into Γ without changing the limit below, but if $r > 1$ the constant is not part of the remainder.

While, for each $0 \leq s \leq t_0$ there exists a positive real number K, such that

$$\left| B\left(\frac{s+1}{(\Phi_s(w)+1)^r} - \frac{1}{\gamma}\right) + \frac{\rho_1(\Phi_s(w))}{(\Phi_s(w)+1)^{1-2r}} \frac{s+1}{(\Phi_s(w)+1)^r}\right| < \frac{K}{r}.$$

So,

$$\begin{aligned} &|(\Phi_t(w)+1)^r - \gamma t - \lambda \log(t+1)| \\ &= \left| \int_0^t \frac{d}{ds}\left((\Phi_s(w)+1)^r - \gamma s - \lambda \log(s+1)\right) ds + (w+1)^r \right| \\ &\leq \left| \int_0^{t_0} \frac{K}{s+1} ds \right| + \left| \int_{t_0}^t \frac{\varepsilon}{s+1} ds \right| + |(w+1)^r| \\ &= \varepsilon \ln(t+1) + (K-\varepsilon)\ln(t_0+1) + |(w+1)^r|. \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{1}{\log(t+1)} |(\Phi_t(w)+1)^r - \gamma t - \lambda \log(t+1)| \\ &\leq \left| \varepsilon + (K-\varepsilon)\frac{\ln(t_0+1)}{\ln(t+1)} + \frac{(w+1)^r}{\ln(t+1)} \right|. \end{aligned}$$

Since, $\varepsilon > 0$ is arbitrary, then (5.8) follows, and

$$(\Phi_t(w)+1)^r = \gamma t + \lambda \log(t+1) + \Gamma_1(w,t) \tag{5.9}$$

with $\lim\limits_{t\to\infty} \dfrac{\Gamma_1(w,t)}{\log(t+1)} = 0$. One can calculate

$$\begin{aligned} \Phi_t(w) + 1 &= \Big(\gamma t + \lambda \log(t+1) + \Gamma_1(w,t)\Big)^{\frac{1}{r}} \\ &= (\gamma t)^{\frac{1}{r}} \left(1 + \frac{\lambda}{\gamma t}\log(t+1) + \frac{\Gamma_1(w,t)}{\gamma t}\right)^{\frac{1}{r}} \\ &= (\gamma t)^{\frac{1}{r}} \left(1 + \frac{\lambda}{\gamma r}\frac{\log(t+1)}{t} + \widetilde{\Gamma}_1(w,t)\right) \\ &= (\gamma t)^{\frac{1}{r}} + \frac{\lambda}{r}(\gamma t)^{\frac{1-r}{r}} \log(t+1) + \Gamma(w,t), \end{aligned}$$

where $\lim\limits_{t\to\infty} \dfrac{t\widetilde{\Gamma}_1(w,t)}{\log(t+1)} = 0$, hence $\lim\limits_{t\to\infty} \dfrac{\Gamma(w,t)}{t^{\frac{1-r}{r}}\log(t+1)} = 0$. This proves formula (5.6). In addition,

$$\lim_{t\to\infty} (t+1)\Big((\Phi_t(w)+1)^r - (\Phi_t(w)+1)^r(1) - \gamma\sigma(w)\Big) =$$

$$\lim_{t\to\infty}(t+1)\cdot\int_1^w \left((\Phi_t(z)+1)^r - \gamma\sigma(z)\right)' dz =$$

$$\lim_{t\to\infty}(t+1)\cdot\int_1^w \frac{r(\Phi_t(z)+1)^{r-1}\phi\left(\Phi_t(z)\right)-\gamma}{\phi(z)} dz.$$

One can calculate

$$\lim_{t\to\infty}(t+1)\left(r(\Phi_t(z)+1)^{r-1}\phi\left(\Phi_t(z)\right)-\gamma\right) =$$

$$= \lim_{t\to\infty}\frac{t+1}{(\Phi_t(z)+1)^r}\left(Br + \frac{\rho_1(\Phi_t(z))r}{(\Phi_t(z)+1)^{1-2r}}\right) = \lambda.$$

So,

$$\lim_{t\to\infty}(t+1)\left((\Phi_t(w)+1)^r - (\Phi_t(1)+1)^r - \gamma\sigma\right) = \lambda\sigma(w).$$

This proves (5.7). ■

We will now show how this proof correlates with our assertions on semigroups over Δ.

5.1 Proof of the main results

In this section we will prove the assertions made in theorem 3.1.
Let $S = \{F_t\}_{t\geq 0}$ be a continuous semigroup of holomorphic self-mappings of the open unit disk Δ and let f be its generator.

Suppose that f admits the representation (3.12), and in the same fashion as before denote $\gamma = 2^{r-1}ar$. We have shown that since f fulfills the requirements set out in (4.6), we can define a semigroup Φ_t in the right half-plane generated by $-\phi$ which is of the form (4.7), a specific case of the form (5.1) which is the one described in theorem 5.1. Recalling our definition of $\Phi_t(w)$ in the right half-plane model,

$$\Phi_t(w) \triangleq C\circ F_t(z)\circ C^{-1}(w),$$

clearly

$$C\circ F_t(z) = \Phi_t\circ C(z). \tag{5.10}$$

Or, more literally,

$$\frac{1+F_t(z)}{1-F_t(z)} = \Phi_t(C(z)).$$

Combining this with (5.2) proven earlier, we see that

$$\frac{1+F_t(z)}{1-F_t(z)} = \Phi_t(C(z)) = (\gamma t)^{\frac{1}{r}} + \Gamma(C(z),t),$$

and since $\lim_{t\to\infty} \dfrac{\Gamma(C(z),t)}{t^{\frac{1}{r}}} = 0$, by defining

$$G(z,t) \triangleq \Gamma(C(z),t) + 1,$$

we get the necessary

$$\lim_{t\to\infty} \frac{G(z,t)}{t^{\frac{1}{r}}} = 0,$$

and (3.13) is proven.

Next, continuing from (5.3), we can use formula (5.10) and the connection between $\sigma(w)$ and h(z), $h(z) = \sigma(C(z))$, to get

$$\lim_{t\to\infty} C(F_t(z))^r - C(F_t(0))^r = \gamma\sigma(C(z)),$$

or

$$\lim_{t\to\infty} \left(\left(\frac{1+F_t(z)}{1-F_t(z)}\right)^r - \left(\frac{1+F_t(0)}{1-F_t(0)}\right)^r \right) = \gamma h(z),$$

which has been put forward earlier as (3.14).

Additionally, since for the same f we can be write ϕ in the same manner as in (5.5), we can apply the same method to (5.9) which was a step in the proof of theorem 5.2 . Now we can see that

$$(1+\Phi_t(C(z)))^r = \left(\frac{2}{1-F_t(z)}\right)^r = (\gamma t) + \lambda \log(t+1) + \Gamma_1(C(z),t),$$

with $\lim_{t\to\infty} \dfrac{\Gamma_1(C(z),t)}{\log(t+1)} = 0$.

Simply by defining $G(z,t) \triangleq \Gamma_1(C(z),t)$ we get (3.16) as an immediate consequence.

Lastly, to prove (3.17), we start with (5.7) in the same theorem. Utilizing the same connections given by Cayley's transform shown above between F and Φ, h and σ we find

$$\begin{aligned} &\lim_{t\to\infty} (t+1)\left((\Phi_t(C(z))+1)^r - (\Phi_t(C(0))+1)^r - \gamma\sigma(C(z))\right) \\ &= \lim_{t\to\infty} (t+1)\left(\left(\frac{2}{1-F_t(z)}\right)^r - \left(\frac{2}{1-F_t(0)}\right)^r - \gamma h(z)\right) \\ &\qquad = \lambda\sigma(C(z)) = \lambda h(z), \end{aligned}$$

which is what we set out to prove.

We have claimed earlier that One of the main results, (3.16), can be re-written in a more pleasing way. We will, as a final word, show this more optimized formula's derivation.
Starting with (3.16), and raising both sides of the equation by a power of $\frac{1}{r}$,

$$\frac{2}{1-F_t(z)} = [\gamma t + \lambda \log(t+1) + G_1(z,t)]^{\frac{1}{r}} = \tag{5.11}$$

$$(\gamma t)^{\frac{1}{r}} \left[1 + \frac{\lambda \log(t+1)}{\gamma t} + \frac{G}{\gamma t}\right]^{\frac{1}{r}} = (\gamma t)^{\frac{1}{r}} \left[1 + \frac{1}{r}\left(\frac{\lambda \log(t+1)}{\gamma t} + \frac{G}{\gamma t}\right) + \Gamma(t)\right]$$

with $\lim_{t\to\infty} \frac{\Gamma(t)}{\left(\frac{\log(t+1)}{t}\right)} = 0.$[3]

Thus, we can re-state (3.16) as we had in (3.18):

$$\frac{2}{1-F_t(z)} = (\gamma t)^{\frac{1}{r}} + \frac{\lambda}{r}(\gamma t)^{\frac{1}{r}-1} \log(t+1) + \Gamma_1(t),$$

with $\lim_{t\to\infty} \frac{\Gamma_1(t)}{\left(\frac{\log(t+1)}{t}\right)} = 0.$ ■

[3]This last step is due to the formula $(1+\alpha)^\beta = 1+\alpha\beta+r(\alpha,\beta)$ with $\lim_{\alpha\to 0} \frac{r(\alpha,\beta)}{\alpha} = 0$

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