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Real-Time Iterative Projection Schemes for Solving the Common Fixed Point Problem and its Applications

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Contents

List of Figures

Abstract

In this project we are concern with the *Common Fixed Point Problem* (CFPP) with demi-contractive operators and its special instance, the *Convex Feasibility Problem* (CFP) in real Hilbert spaces. Motivated by the recent result of Ordoñez et al. [35] and in general, the field of real-time/online algorithms [31, 21, 20], in which the entire input is not available from the beginning and given piece-by-piece, we propose an online block-iterative scheme for solving CFPPs and CFPs in which the involved operators/sets emerge along time. This scheme is capable of operating on any block, for any finite number of iterations, before moving, in a serial way, to the next block.

The scheme is based on the recent novel result of Reich and Zalas [37] known as the *Modular String Averaging* (MSA) procedure. Convergence of the scheme is then follows [37] as well as classical results in the fields of fixed point theory and variational inequalities, such as [34]).

Numerical experiments for linear and non-linear feasibility problems with applications to image recovery are presented and demonstrate the validity and the potential applicability of the new scheme, which can be used for online scenarios, for example.

1 Introduction

In this project we are concern with the *Common Fixed Point Problem* (CFPP) and its special instance, the *Convex Feasibility Problem* (CFP) in real Hilbert spaces $\mathcal{H}$ (with the inner product $\langle ., . \rangle$, and the induced norm $\|.\|$). Given operators $U_i : \mathcal{H} \to \mathcal{H}$, for $i \in I := \{1, 2, \ldots, m\}$, with non-empty fixed points sets, the common fixed points problem consists of finding a point $x^* \in \mathcal{H}$ such that

$$x^* \in \cap_{i=1}^{m} \operatorname{Fix}(U_i). \tag{1.1}$$

In the convex feasibility problem, we are given m non-empty, closed and convex sets $C_i \subseteq \mathcal{H}$ for $i \in I$. The problem is then formulated as finding a point $x^* \in \mathcal{H}$ such that

$$x^* \in \cap_{i=1}^{m} C_i \neq \emptyset. \tag{1.2}$$

It is clear that if we choose $U_i = P_{C_i}$ for all $i \in I$, where P_{C_i} denotes the orthogonal projection onto the i-th set C_i (will be explained further) in the CFPP (1.1), then the CFP (1.2) is obtained.

The CFPP and the CFP serve as essential modelling tools which stand at the core of many significant real-world problems, for example in imaging, sensor networks, radiation therapy treatment planning, resolution enhancement and in many others; see e.g., [14, 5]. One of the earliest iterative procedure for solving CFPPs, see e.g., [34], has the following general form: choose an arbitrary starting point $x^0 \in \mathcal{H}$

$$x^{k+1} = T(x^k) \tag{1.3}$$

where the operator $T : \mathcal{H} \to \mathcal{H}$ is fixed and depends on the family of operators $\{U_i : \mathcal{H} \to \mathcal{H} \mid i \in I\}$. A more general fixed point scheme allows to include a family of operators $\{T_k : \mathcal{H} \to \mathcal{H}\}_{k=0}^{\infty}$; for example see the generalized Opial method [11, Section 3.6]. The iterative procedure is formulated as follows: choose an arbitrary starting point $x^0 \in \mathcal{H}$

$$x^{k+1} = T_k(x^k) \tag{1.4}$$

where the family of operators $\{T_k\}_{k=0}^{\infty}$ depends on $\{U_i\}_{i \in I}$ and could have different algorithmic structures, such as

1. **Cyclic (with relaxation):** $\alpha_k \in [\varepsilon, 2-\varepsilon]$, for $\varepsilon > 0$: $T_k = U_{i(k)}$, where $i(k) = (k \bmod m) + 1$;

2. **Simultaneous:** $T_k := \frac{1}{m}\sum_{i=1}^{m} U_i$;

3. **Composition:** $T_k := \prod_{i=1}^{m} U_i$.

4. **Greedy (remotest-set):** $T_k := U_{i_k}$, where $i_k = \operatorname{argmax}_{i\in I} \operatorname{dist}(\cdot, \operatorname{Fix}(U_i))$; where $\operatorname{dist}(\cdot,\cdot)$ is the `distance function` between a point and a set.

Going back to the Convex Feasibility Problem (CFP), we wish to focus on the class of projection methods. In the 1930s Kaczmarz [30] and Cimmino [17] introduced iterative projection methods for solving systems of linear inequalities $Ax \le b$, where $A \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. It appears that this problem can be easily converted to an equivalent CFP by the following: denote by A^i and b_i the i-th row and entry of A and b, respectively. Define the set (half-space):

$$H_i^- := \{z \in \mathbb{R}^n \mid \langle A^i, z\rangle \le b_i\} \tag{1.5}$$

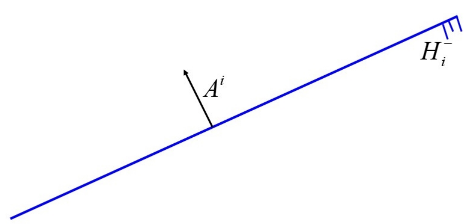

Figure 1: Geometric interpretation of a Half-space H_i^-.

and then we obtain:

$$Ax \le b \Leftrightarrow x \in \cap_{i=1}^{m} H_i^- \tag{1.6}$$

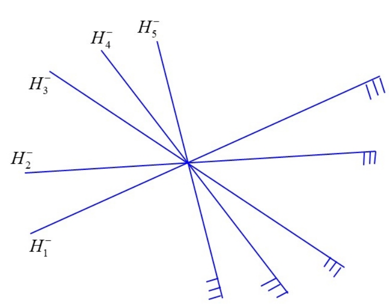

Figure 2: Geometric interpretation of the intersection of half-spaces

Kaczmarz [30] and Cimmino [17] methods use orthogonal projections onto the half-spaces H_i^- in a sequential and simultaneous way, respectively. This is in general the characterization of the class of projection methods, which are iterative procedures which use projections, of different types, onto sets by taking into account that the projection onto the intersection of the sets is a very hard computational task, while projections onto the individual sets are relatively easier. This is the reason why these methods are applied successfully in many real-world applications and were called "Swiss Army knives", see [6]. Since the introduction of Kaczmarz [30] and Cimmino [17], the class of projection methods was developed intensively and is capable of solving the general convex feasibility problem (1.2) and it also include various algorithmic structures such as sequential, simultaneous, block-iterative, string-averaging and more, see [14] as well as [10, 11, 16, 22, 23]).

In the recent paper of Ordoñez et al. [35], two real-time projection methods (the Diagonally Relaxed Orthogonal Projections (DROP) [1] and the Component-Averaged Row Projections (CARP) [25]) are introduced for solving huge, sparse and overestimated system of linear equations $Ax = b$ where $A \in \mathbb{R}^{m\times n}$ with $m \sim 10^3$ and $n \sim 10^9$, arising in the area of proton computed tomography (pCT). While iterative projection methods are applied successfully for solving big and sparse problems, see for example [36], in [35] it is shown experimentally that the real-time DROP and CARP preform much "better "and "faster".

So, following [35] and in general the filed of online algorithms [31, 21, 20], our goal in this project is to introduce a new fixed point iteration of type (1.3) or (1.4) designed for the CFPP or a projection method for the CFP. We focus on the case where the entire input (operators/sets) is not available from the beginning and given piece-by-piece, this calls for an online block-iterative scheme for solving CFPPs and CFPs which is capable of operating on segments of input and incorporate new input when it emerges. In the CFPP, the idea is that the operators U_i for $i \in I$ are provided in blocks $I = I_1 \cup I_2 \cup \ldots \cup I_M$, $1 \leq M \leq m$ and successively in time. We adopt the Reich and Zalas [37] *Modular String Averaging* (MSA) procedure and show how this can be used for the above scenarios. Numerical experiments show the potential applicability and advantages of the proposed method to online linear and non-linear feasibility problems.

2 Preliminaries

Throughout this project $\mathcal{H}$ is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. We write $x^k \rightharpoonup x$ and x and $x^k \to x$ to indicate that the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ converges weakly and strongly to x, respectively.

We now recall some definitions and properties of several classes of operators. These and more can be found, for example in the excellent of Cegielski [11].

Definition 2.1 *Let $U : \mathcal{H} \to \mathcal{H}$ be some operator.*

- *The **`fixed point set`** of U, denoted by* $\mathrm{Fix}(U)$ *is defined as*

$$\mathrm{Fix}(U) := \{x \in \mathcal{H} \mid U(x) = x\}. \tag{2.1}$$

- *The operator U is called **`cutter`** if for all $x \in \mathcal{H}$ and all $z \in \mathrm{Fix}(U)$,*

$$\langle z - U(x), x - U(x) \rangle \leq 0. \tag{2.2}$$

The set of fixed points of a cutter U is closed and convex set and moreover, $\mathrm{Fix}(U) = \bigcap_{x \in \mathcal{H}} H(x, U(x))$, where $H(x, U(x)) := \{z \in \mathcal{H} \mid \langle z - U(x), x - U(x) \rangle \leq 0\}$; see [4, Proposition 2.6(ii)] and Figure 3 for the geometric interpretation of a cutter.

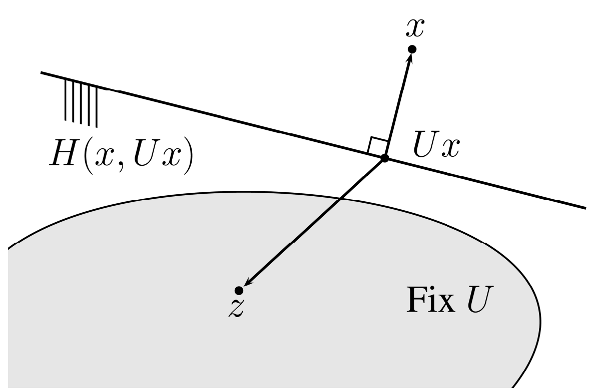

Figure 3: Geometric interpretation of a cutter U.

- *The operator U is called **`non-expansive`** (NE) if for all $x, y \in \mathcal{H}$,*

$$\|U(x) - U(y)\| \leq \|x - y\|. \tag{2.3}$$

- *The operator U is called* ***firmly non-expansive*** *(FNE) [18], if for all $x, y \in \mathcal{H}$,*

$$\|U(x) - U(y)\|^2 \leq \|x - y\|^2 - \|(U(x) - x) - (U(y) - y)\|^2. \qquad (2.4)$$

  An FNE operator is a cutter [11, Theorem 2.2.5].

- *The operator U is called* ***quasi-nonexpansive*** *(QNE), if for all $(x, q) \in \mathcal{H} \times \operatorname{Fix}(U)$,*

$$\|U(x) - q\| \leq \|x - q\|. \qquad (2.5)$$

- *The operator U with $\operatorname{Fix}(U) \neq \emptyset$ is called ρ-**demi-contractive** (see for example [19]), where $\rho \in [-1, 0)$, if for all $(x, z) \in \mathcal{H} \times \operatorname{Fix}(U)$*

$$\|U(x) - z\|^2 \leq \|x - z\|^2 - \rho\|U(x) - x\|^2 \qquad (2.6)$$

- *The operator U is called ρ-**strongly quasi-nonexpansive**, where $\rho \geq 0$, if for all $(x, z) \in \mathcal{H} \times \operatorname{Fix}(U)$.*

$$\|U(x) - z\|^2 \leq \|x - z\|^2 - \rho\|U(x) - x\|^2 \qquad (2.7)$$

  If $\rho > 0$, then U is called ***strongly quasi-nonexpansive****.*

- *For $\alpha \in [0, \infty]$, the operator $U_\alpha := Id + \alpha(U - Id)$ is called an α-**relaxation of** U, α is called a* ***relaxation parameter****. It is easy to see that for every $\alpha \neq 0$,*

$$\operatorname{Fix}(U) = \operatorname{Fix}(U_\alpha). \qquad (2.8)$$

- *The operator U is called* ***averaged*** *[2] (also [9]) if there exists a non-expansive operator $N : \mathcal{H} \to \mathcal{H}$ and a number $c \in (0, 1)$ such that*

$$U = (1 - c)Id + cN \qquad (2.9)$$

  where Id is the identity operator on $\mathcal{H}$.

- *A quasi-nonexpansive operator U is called* ***demi-closed*** *at a point $y \in \mathcal{H}$, if for any sequence $\{x^k\}_{k=0}^{\infty} \subset \mathcal{H}$ we have*

$$\left.\begin{array}{l} x^k \rightharpoonup \overline{x} \\ U(x^k) \to y \end{array}\right\} \Longrightarrow U(\overline{x}) = y. \qquad (2.10)$$

- *A quasi-nonexpansive operator U is called* ***approximately shrinking*** *if for any bounded sequence $\{x^k\}_{k=0}^{\infty} \subseteq \mathcal{H}$, we have*

$$\lim_{k\to\infty} \|U(x^k) - x^k\| = 0 \quad \Longrightarrow \quad \lim_{k\to\infty} \operatorname{dist}(x^k, \operatorname{Fix}(U)) = 0 \tag{2.11}$$

  For more details regarding this class of operators see for example [12].

A useful result showing the relation between two classes of operators from above and is relevant to our analysis is the following theorem. The proof can be found for example in [11, Theorem 2.1.39] and presented here for the convenient of the reader.

Theorem 2.2 *Let $U : \mathcal{H} \to \mathcal{H}$ be an operator having a fixed point and let $\alpha \in (0, 2]$. Then U is a cutter if and only if its U_α is $(2-\alpha)/\alpha$-strongly quasi-nonexpansive.*

Proof. From the definition of α-strongly quasi-nonexpansive, we get for all $(x, z) \in \mathcal{H} \times \operatorname{Fix}(U)$, that

$$\|U_\alpha(x) - z\|^2 \leq \|x - z\|^2 - \alpha\|U_\alpha(x) - x\|^2. \tag{2.12}$$

Now, with α is $(2-\alpha)/\alpha$

$$\|U_\alpha(x) - z\|^2 \leq \|x - z\|^2 - \frac{(2-\alpha)}{\alpha}\|U_\alpha(x) - x\|^2. \tag{2.13}$$

Since $U_\alpha(x) - x = \alpha(U(x) - x)$, the inner product properties yield

$$\begin{aligned}
&\|U_\alpha(x) - z\|^2 - \|x - z\|^2 + \frac{(2-\alpha)}{\alpha}\|U_\alpha(x) - x\|^2 \\
&= \|x - z + \alpha(U(x) - x)\|^2 - \|x - z\|^2 + \alpha(2-\alpha)\|U(x) - x\|^2 \\
&= 2\alpha(\|U(x) - x\|^2 - \langle z - x, U(x) - x\rangle) \\
&= 2\alpha\langle z - U(x), x - U(x)\rangle)
\end{aligned} \tag{2.14}$$

for all $(x, z) \in \mathcal{H} \times \operatorname{Fix}(U)$. And the desired result is obtained. ■

The next principle is known as the Demiclosedness Principle [8].

Demiclosedness Principle. Let $\mathcal{H}$ be a real Hilbert space, $C \subseteq \mathcal{H}$ a closed and convex set, and let $S : C \to \mathcal{H}$ be a non-expansive mapping; then

$Id - S$ is demi-closed at $y \in \mathcal{H}$.

Another useful result needed for our analysis is presented next, introduced in [13, Proposition 4.1].

Proposition 2.3 *Let $U : \mathcal{H} \to \mathcal{H}$ be a quasi-nonexpansive operator. Then the following assertions hold:*

1. *If U is approximately shrinking, then $U - Id$ is demi-closed at 0;*
2. *If $\dim(\mathcal{H}) < \infty$ ($\mathcal{H}$ is finite dimensional) and $U - Id$ is demi-closed at 0, then U is approximately shrinking.*

Definition 2.4 *Let $C_i \subseteq \mathcal{H}$, for $i \in I$, be closed and convex sets with a nonempty intersection $\mathcal{C} := \cap_{i \in I} C_i \neq \emptyset$. We say that the family of sets $\mathfrak{C} := \{C_i \mid i \in I\}$ is* **boundedly regular** *if for any bounded sequence $\{x^k\}_{k=0}^{\infty} \subseteq \mathcal{H}$, we have*

$$\lim_{k \to \infty} \max_{i \in I} \operatorname{dist}(x^k, C_i) = 0 \quad \Longrightarrow \quad \lim_{k \to \infty} \operatorname{dist}(x^k, \mathcal{C}) = 0. \tag{2.15}$$

The next proposition, taken from [3, Proposition 5.4 (iii), Corollary 5.14, Corollary 5.22], present conditions which guarantee that a family of sets is boundedly regular.

Proposition 2.5 *Let $C_i \subseteq \mathcal{H}$, for $i \in I$, be closed and convex sets with a nonempty intersection $\mathcal{C} := \cap_{i=1}^{m} C_i$. If one of the following conditions hold:*

1. $\dim(\mathcal{H}) < \infty$;
2. $\operatorname{int}(\mathfrak{C}) \neq \emptyset$;
3. *Each C_i is a half-space.*

Then the family of sets $\mathfrak{C} := \{C_i \mid i \in I\}$ is boundedly regular.

Now it is time to recall the metric projection onto a closed and convex set. Let $C \subset \mathcal{H}$. For each point $x \in \mathcal{H}$, there exists a unique nearest point in C, denoted by $P_C(x)$ and such that

$$\|x - P_C(x)\| \leq \|x - y\| \text{ for all } y \in C. \tag{2.16}$$

The mapping $P_C : \mathcal{H} \to C$ is called the *metric projection* of $\mathcal{H}$ onto C and it is a non-expansive mapping of $\mathcal{H}$ onto C (actually FNE (hence a cutter) see [11, Theorem 2.2.21]). P_C is characterized [24, Section 3] by the following two properties:

$$P_C(x) \in C \tag{2.17}$$

(hence $\mathrm{Fix}(P_C) = C$) and

$$\langle x - P_C(x), P_C(x) - y\rangle \geq 0 \text{ for all } x \in \mathcal{H},\ y \in C, \tag{2.18}$$

and if C is a hyper-plane, then (2.18) becomes an equality. It follows that

$$\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2 \text{ for all } x \in \mathcal{H},\ y \in C. \tag{2.19}$$

Another important type of projection is the next subgradient projection, which is also a cutter. This projection is very useful when the convex set C has a sub-level set representation of a convex function $g : \mathcal{H} \to \mathbb{R}$, that is $C := \{x \in \mathcal{H} \mid g(x) \leq 0\}$.

Example 2.6 *Let $f : \mathcal{H} \to \mathbb{R}$ be a convex and continuous function with a nonempty sub-level set $S(f, 0) := \{x \mid f(x) \leq 0\}$. Denote by $\partial f(x)$ its* ***subdifferential****, that is, $\partial f(x) := \{g \in \mathcal{H} \mid f(y) - f(x) \geq \langle g, y - x\rangle$ for all $y \in \mathcal{H}\}$. By the continuity of f, the set $\partial f(x) \neq \emptyset$ for all $x \in \mathcal{H}$ (see [5, Proposition 16.3 and Proposition 16.14]). For each $x \in \mathcal{H}$, let $g_f(x) \in \partial f(x)$ be a given* ***subgradient****. The so-called* ***subgradient projection*** *relative to f is the operator $P_f : \mathcal{H} \to \mathcal{H}$ defined by*

$$P_f(x) := \begin{cases} x - \frac{f(x)}{\|g_f(x)\|^2} g_f(x) & \text{if } g_f(x) \neq 0, \\ x & \text{otherwise.} \end{cases} \tag{2.20}$$

One can verify that $\mathrm{Fix}(P_f) = S(f, 0)$, see for example [11, Lemma 4.2.5], and that P_f is a cutter, see [11, Corollary 4.2.6].

Several "simple"sets in which the orthogonal projection onto them has a closed formula, are presented next.

Example 2.7 *1. Let $a \in \mathbb{R}^n$ (non-zero) and $\beta \in \mathbb{R}$. The projection onto* ***half-space*** *$H^- = \{z \in \mathbb{R}^n \mid \langle a, z\rangle \leq \beta\}$ is given as following.*

$$P_{H^-}(x) = \begin{cases} x & \text{if } \langle a, x\rangle \leq \beta \\ x - \frac{\langle x, a\rangle - \beta}{\|a\|^2} a & \text{if } \langle a, x\rangle > \beta. \end{cases} \tag{2.21}$$

In case we have equality in H^-, that is $H = \{z \in \mathbb{R}^n \mid \langle a, z\rangle = \beta\}$, then the set is called ***hyper-plane****,*

2. *Consider the* **non-negative orthant** $\mathbb{R}^n_+$*, the projection onto it is defined as:*

$$\left[P_{\mathbb{R}^n_+}(x)\right]_i = \left\{ \begin{array}{ll} x_i & \textit{if } x_i \geq 0 \\ 0 & \textit{else.} \end{array} \right. \tag{2.22}$$

3. *Consider the* **closed ball** $B(z,r) := \{x \in \mathbb{R}^n \mid \|x - z\| \leq r\}$*, where* $z \in \mathbb{R}^n$ *and* $r > 0$*. The projection onto the ball is defined as:*

$$P_{B(z,r)}(x) = \left\{ \begin{array}{ll} x & \textit{if } \|x - z\| \leq r \\ z + \frac{r}{\|x-z\|}(x - z) & \textit{if } \|x - z\| > r. \end{array} \right. \tag{2.23}$$

Now that we have defined different types of projections as well as classes of operators, we wish to recall two special types of projection methods: sequential (Kaczmarz, also known as *Successive Orthogonal Projections* (SOP) [27], *Projections Onto Convex Sets* (POCS) [3] and *Algebraic Reconstruction Technique* (ART) [26] for the linear case) and block-type iterative methods (fully simultaneous if there is only one block, in the linear case it reduces to Cimmino method [17]). For this purpose, we are focus on the convex feasibility problem with non-empty, closed and convex sets $C_i \subseteq \mathcal{H}$ for $i \in I$. The next definitions of control sequences, $\{i(\nu)\}_{\nu=0}^{\infty}$, determine the ordering in which the orthogonal-projections onto the sets C_i, $i \in I$ are involved and hence determine the structure of the algorithm.

Definition 2.8 1. *The sequence* $\{i(\nu)\}_{\nu=0}^{\infty}$ *is called* **cyclic control***, if* $i(\nu) = (\nu \bmod)\, m + 1$*, where* m *is the number of the sets in (1.2).*

2. *The sequence* $\{i(\nu)\}_{\nu=0}^{\infty}$ *is called* **almost cyclic control** *on* $I = \{1, 2, \cdots, m\}$ *if* $i(\nu) \in I$*, for all* $\nu \geq 0$*, and there exists an integer* $Q \geq m$ *(called the* **almost cyclically constant***) such that*

$$I \subseteq \{i(\nu+1), i(\nu+2), \cdots, i(\nu+Q)\} \textit{ for all } \nu \geq 0. \tag{2.24}$$

3. *The sequence* $\{i(\nu)\}_{\nu=0}^{\infty}$ *is called* **remotest set control** *if is obtained by determining* $i(\nu)$ *such that*

$$\mathrm{dist}(x^\nu, C_{i(\nu)}) = \max\{\mathrm{dist}(x^\nu, C_i) \mid i \in I\}. \tag{2.25}$$

4. *The sequence* $\{i(\nu)\}_{\nu=0}^{\infty}$ *is called* **random control** *if* $i(\nu) \in I$ *is chosen randomly and independently determined according to a fixed probability distribution* $\{p_i\}$

Now we present the sequential and simultaneous projection methods for solving CFPs.

Algorithm 2.9 ***SOP method***

Initialization: *Let $x^0 \in \mathcal{H}$ be arbitrary starting point.*

Iterative step: *Given the current iterate x^k, compute the next iterate by*

$$x^{k+1} = x^k + \lambda_k \left(P_{C_{i(\nu)}}(x^k) - x^k \right) \tag{2.26}$$

where $P_{C_{i(\nu)}}$ stands for the orthogonal projection onto the set $C_{i(\nu)}$, $\lambda_k \in [\varepsilon_1, 2 - \varepsilon_2]$ for all $k \geq 0$ and $\varepsilon_1, \varepsilon_2 > 0$. The control sequence $\{i(\nu)\}_{\nu=0}^{\infty}$ is cyclic on I.

Now for the next algorithm we need to define the following terms. A vector $\omega = (\omega(i))_{i \in I}$ is called *weight vector* when $\omega(i) \geq 0$ for all $i \in I$ and $\sum_{i \in I} \omega(i) = 1$. Given a weight vector ω, we can define the *convex combination* $P_\omega(x) := \sum_{i \in I} \omega(i) P_{C_i}$. A sequence of weight vectors $\{\omega^k\}_{k=0}^{\infty}$ is called *fair* if for any $i \in I$ there exist infinitely many values of k for which $\omega^k(i) > 0$.

Algorithm 2.10 ***Block-type method***

Initialization: *Let $x^0 \in \mathcal{H}$ be arbitrary starting point.*

Iterative step: *Given the current iterate x^k, compute the next iterate by*

$$x^{k+1} = x^k + \lambda_k \left(P_{\omega^k}(x^k) - x^k \right) \tag{2.27}$$

where $\{\omega^k\}_{k=0}^{\infty}$ is a fair sequence of weight vectors and $\lambda_k \in [\varepsilon_1, 2 - \varepsilon_2]$ for all $k \geq 0$ and $\varepsilon_1, \varepsilon_2 > 0$.

For the illustration of several types of projection methods for solving the convex feasibility problem, we restrict ourself to the linear feasibility problem, which is the system of linear equations $Ax = b$, where $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. Denoting by A^i and b_i the i-th row and entry of A and b, respectively, and define the i-th hyper-plane $H_i = \{z \in \mathbb{R}^n \mid \langle A^i, z \rangle = b_i\}$. Illustrations of these and other projection methods is given in Figure 4, reproduced from [15].

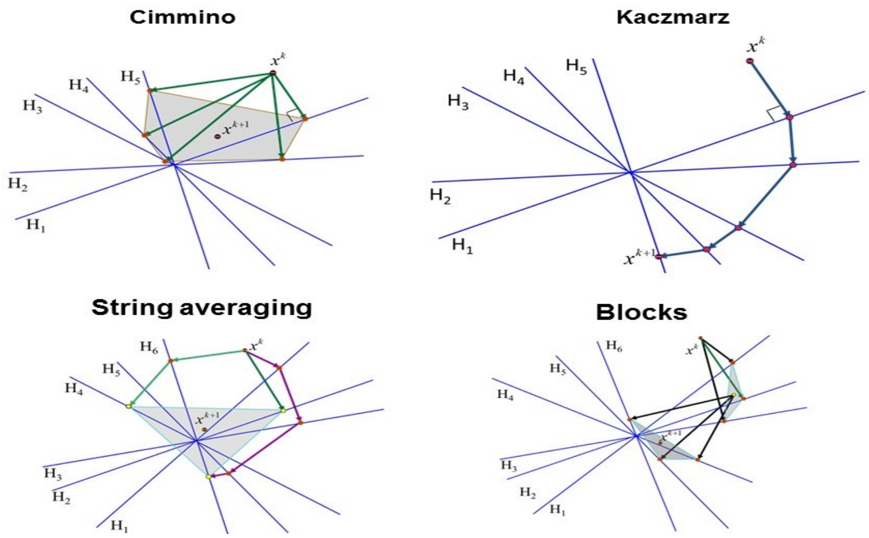

Figure 4: Different projection methods for the linear case. The figure is reproduced from [15]

Remark 2.11 *Observe that the fixed point iterations (1.3) and (1.4) include the above methods. For example, if we consider the common fixed point problem with $U_i = P_{C_i}$, then we obtain the convex feasibility problem. Moreover, if $T_k = U_{i(k)}$, where $i(k) = (k \bmod) m+1$ we obtain the SOP method (2.9) and if we consider only one block I of size m, then by taking $T = \frac{1}{m}\sum_{i=1}^{m} U_i$ we obtain Cimmino method for the linear case and block-type method in general.*

Next we recall two fixed point theorems, the classical Opial Theorem [34] and its generalization [11, Section 3.6].

Theorem 2.12 *Let $\mathcal{H}$ be a real Hilbert space and let $C \subset \mathcal{H}$ be closed and convex set. If $T : C \to C$ is an averaged operator with $\operatorname{Fix}(T) \neq \emptyset$ then, for any $x^0 \in C$, the sequence $\left\{x^k\right\}_{k=0}^{\infty}$, generated by $x^{k+1} = T(x^k)$, converges weakly to a point $x^* \in \operatorname{Fix}(T)$.*

Next is the generalized Opial's theorem, see for example [11, Section 3.6], which is designed to handel a family of operators $\{T_k : \mathcal{H} \to \mathcal{H}\}_{k=1}^{\infty}$.

Theorem 2.13 *Let $C \subseteq \mathcal{H}$ be nonempty, closed and convex set, $S : C \to \mathcal{H}$ be an operator with a fixed point set and such that $S - Id$ is demi-closed at 0. Let $\{T_k : \mathcal{H} \to \mathcal{H}\}_{k=1}^{\infty}$ be an asymptotically regular sequence of quasi-nonexpansive operators such that $\mathrm{Fix}(S) \subseteq (\cap_{k=1}^{\infty} \mathrm{Fix}(T_k))$. Let the sequence $\left\{x^k\right\}_{k=0}^{\infty}$, generated by $x^{k+1} = T_k(x^k)$, with an arbitrary $x^0 \in \mathcal{H}$.*

1. *If the sequence of operators $\{T_k\}_{k=1}^{\infty}$ has the property*

$$\lim_{k\to\infty} \|T_k(x^k) - x^k\| = 0 \quad \Longrightarrow \quad \lim_{k\to\infty} \|S(x^k) - x^k\| = 0, \tag{2.28}$$

then $\left\{x^k\right\}_{k=0}^{\infty}$ converges weakly to a point $\mathrm{Fix}(S)$.

2. *If $\mathcal{H}$ is finite dimensional and the sequence of operators $\{T_k\}_{k=1}^{\infty}$ has the property*

$$\lim_{k\to\infty} \|T_k(x^k) - x^k\| = 0 \quad \Longrightarrow \quad \liminf_{k\to\infty} \|S(x^k) - x^k\| = 0, \tag{2.29}$$

then $\left\{x^k\right\}_{k=0}^{\infty}$ converges to a point $\mathrm{Fix}(S)$.

3 The Algorithm

In this chapter we focus on the common fixed point problem (CFPP) (1.1) with the family of demi-contractive operators $\{U_i\}_{i\in I}$, such that $\cap_{i\in I}\operatorname{Fix}(U_i) \neq \emptyset$. The situation is then that the indices set I is decomposed into M blocks $I = I_1 \cup \cdots \cup I_M$ by choosing $\{m_t\}_{t=0}^{M} \subset \mathbb{Z}$, ($\mathbb{Z}$ is the integer set)such that $0 = m_0 < m_1 < \ldots < m_M = m$ and for each $1 \leq t \leq M$, the subset $I_t := \{m_{t-1}+1, m_{t-1}+2, \cdots, m_t\}$. This of course divides the family of operators $\{U_i\}_{i\in I}$ into corresponding groups of operators. Since our concern is to introduce an online block-iterative scheme, we focus on the case where the blocks, and the corresponding operators, are not given from the beginning, but provided in time, in a serial way. In the recent paper of Ordoñez et al. [35], two real-time projection methods ((DROP) [1] and (CARP) [25]) for solving systems of linear equation $Ax = b$ where $A \in \mathbb{R}^{m\times n}$ with $m \sim 10^3$ and $n \sim 10^9$, arising in the area of proton computed tomography (pCT). Recently, Reich and Zalas [37] introduced the *Modular String Averaging* (MSA) procedure for solving the common fixed point problem in real Hilbert spaces. Their scheme is very flexible and allows to construct intermediate operators T_k, called *modules*, which can be involved in a inner loop of a wider algorithm with a finite number of iterations N_k. Our observation is that this can procedure can be adopted for our needs along with convergence proof, which [35] is missing.

For the algorithm's representation we list several structures of the operators T_k, constructed by the family of operators $\{U_i\}_{i\in I}$ with respect to the M blocks $I = I_1 \cup \cdots \cup I_M$, and are involved in an intermediate loop of our algorithm. This structures are presented as special cases for [37, Modular String Averaging], for more details as well as intensive historical review see [37] and the many references therein.

Definition 3.1

1. ***Cyclic (with relaxation):*** $\alpha_k \in [\varepsilon, 2-\varepsilon]$, *for* $\varepsilon > 0$: $T_k = U_{i(k)}$, *where* $i(k) = (k \bmod m) + 1$;

2. ***Convex combination:*** *For a weight vector* $\omega^k(i) \geq 0$ *for all* $i \in I_k$ *and* $\sum_{i\in I_k}\omega^k(i) = 1$, *let* $T_k = \sum_{i\in I_k}\omega^k(i)U_i$;

3. ***Composition:*** $T_k = \prod_{i\in I_k} U_i$.

4. ***Blocks:*** $\alpha_k \in [\varepsilon, 2-\varepsilon]$, *for* $\varepsilon > 0$: $T_k = Id + \alpha_k\left(\sum_{i\in I_k}\omega^k(i)U_i - Id\right)$;

5. ***Greedy (remotest-set):*** *$T_k := U_{i_k}$, where $i_k = \operatorname{argmax}_{i \in I_k} \operatorname{dist}(\cdot, \operatorname{Fix}(U_i))$;*

More special structures that can be used are *string averaging* as well as various types of *Douglas-Rachford operators* (see for example [7]) in case that $2U_iId$ are used instead of U_i.

Algorithm 3.2 ***Online block-iterative scheme***

Initialization*: Let $x^0 \in \mathcal{H}$ be arbitrary starting point, define $N_0 \in \mathbb{N}$ (number of iterations) and given the first block I_1 and its corresponding subset of operators $\{U_i\}_{i \in I_1}$. Compute x^1 via*

$$x^1 = T_0(x^0) \tag{3.1}$$

where the operator T_0 can be constructed according to Definition 3.1, that is cyclic, simultaneous or composition of $\{U_i\}_{i \in I_1}$.

Iterative step*: Given the current iterate x^k, define $N_k \in \mathbb{N}$ (number of iterations) compute the next iterate by*

$$x^{k+1} = T_k(x^k) \tag{3.2}$$

where the operator T_k can be constructed as follows.

1. *If $k < m$: the blocks $I_1, I_2, \ldots, I_k$ are given and hence the operators $\{U_i\}_{i \in I_1 \cup \cdots \cup I_k}$. Then T_k can be constructed with respect to each, some or all the operators $\{U_i\}_{i \in I_1 \cup \cdots \cup I_k}$, in a cyclic, simultaneous or composition way, based on Definition 3.1.*

2. *If $k \geq m$: then all the blocks are given and hence also the operators $\{U_i\}_{i \in I}$ and then T_k can be constructed based on Definition 3.1, with respect to the all family of operators $\{U_i\}_{i \in I}$.*

3.1 Convergence

For the convergence of our Algorithm 3.2 we assume that the following conditions hold.

Condition 3.3 *The operators U_i for all $i \in I$ are demi-contractive with* $\operatorname{Fix}(U_i) \neq \emptyset$ *and such that U_i, α (α-relaxation of U_i) is $(2-\alpha)/\alpha$-strongly quasi-nonexpansive.*

Condition 3.4 $I \subseteq I_k \cup I_{k+1} \cup \cdots \cup I_{k+s-1}$ *for each $k = 0, 1 \ldots$ and some $s \geq m-1$.*

Condition 3.5 *The sequence $\{N_k\}_{k=0}^{\infty}$, which is the number of iterations per each block, is bounded.*

Reich and Zalas [37] proposed in *Numerical Algorithm* the *Modular String Averaging* (MSA) procedure for solving the common fixed point problem in real Hilbert spaces. They introduced a flexible procedure [37, Procedure 1.1]) for constructing intermediate operators T_k, called *modules*, which can be involved in a inner loop of a wider algorithm with a finite number of iterations N_k, with a family of operators $\{U_i\}_{i\in I}$. Due to the modularity of their scheme, and by assuming Conditions 3.3-3.5, the convergence of our online block-iterative scheme, Algorithm 3.2, follows directly from the proof of Theorem 4.1 of Reich and Zalas [37], although there the online term is not mentioned. The next theorem is a modification of [37, Theorem 4.1] adjusted to Algorithm 3.2.

Theorem 3.6 *Let $\mathcal{H}$ be a real Hilbert space and given operators $U_i : \mathcal{H} \to \mathcal{H}$ for $i \in I$, such that* $\operatorname{Fix}(U_i) \neq \emptyset$*. Assume that Conditions 3.3-3.5 hold and let the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ be generated by Algorithm 3.2.*

1. *If for each $i \in I$, the operator U_i satisfies Opials demi-closedness principle, then the sequence $\{x^k\}_{k=0}^{\infty}$ converges weakly to some point in $\mathcal{C} = \cap_{i=1}^{m} \operatorname{Fix}(U_i)$.*

2. *If for each $i \in I$, the operator U_i is approximately shrinking and the family $\mathfrak{C} := \{\operatorname{Fix}(U_i) \mid i \in I\}$ is boundedly regular, then the sequence $\{x^k\}_{k=0}^{\infty}$ converges strongly to some point in $\mathcal{C}$.*

It worth mentioning that in this project we assume that U_i for all $i \in I$ are demi-contractive and hence by Theorem 2.2, for all $i \in I$ and $\alpha \in (0, 2]$ we define U_i, α (α-relaxation of U_i) such that it is $(2 - \alpha)/\alpha$-strongly quasi-nonexpansive and thus a cutter, and this is what is used in Algorithm 3.2.

Remark 3.7

1. *Following Condition 3.3 and Theorem 2.2 we get that U_i are cutters and also the structure of Algorithm 3.2 can arbitrary and of any type based on Definition 3.1.*
2. *Condition 3.4 means that the control sequence is almost cyclic (Definition 2.8 (ii)).*
3. *Condition 3.5 means that the number of iterations N_k is bounded, which means that any finite number of intermediate steps within each blocks is valid.*
4. *In case that $U_i = P_{C_i}$ and $\mathcal{H} = \mathbb{R}^n$, then all continuity related assumptions such as cutter, demi-closed, approximately shrinking are satisfied.*
5. *The random control sequence for feasibility problems and in particular for linear feasibility problems (known as randomized Kaczmarz method, see e.g., [33, 32]) appeared to be an efficient control sequence. Although the convergence proof does not cover this situation, it is interesting to investigate its theoretical behaviour, and hence we plan to define it as for our future work. Despite this, we included it in our numerical experiments. Another related result in which random control sequences are also considered is the class of stochastic algorithms, in particular for feasibility problems and variational inequalities, see the work of Iusem et. al. [29].*
6. *Ordoñez et al. [35] presented two real-time projection methods (DROP) [1] and (CARP) [25] for solving huge, sparse and overestimated system of linear equations $Ax = b$ where $A \in \mathbb{R}^{m\times n}$ with $m \sim 10^3$ and $n \sim 10^9$, arising in the area of proton computed tomography (pCT). Their paper reports very promising experimental behavior but unfortunately is missing the mathematical theory. Although our approach does not covers their schemes, we provide are able to provide an analysis for a flexible scheme which can be applied not only to linear feasibility problems.*

4 Numerical experiments

In this chapter we compare 4 variants of our on line scheme: Cimmino [17] (Algorithm 2.9), Kaczmarz [30] (Algorithm 2.10), Randomized Kaczmarz ([33, 32]) and Greedy-Kaczmarz for linear and non-linear (quadratic) CFPs in Euclidean spaces $\mathbb{R}^d$. All the numerical results are completed on a standard Lenovo laptop with Intel(R) Core(TM) i5-4200MQ CPU 1.6GHz with 8 GB memory. The programme is implemented in MATLAB 2017b.

Example 4.1 ***Linear CFP:*** *In this example consider solving a system of linear equations $Ax = b$, in particular, the reconstruction of a test image $x \in [0,1]^n$ (Lenna) from a limited number of tomographic projections. Each pixel is denoted by $x_i \in [0,1]$ and each entry in $b \in \mathbb{R}^m$, called tomographic measurement or single projection, corresponds to the integrated gray values of x along the single ray. Each matrix entry $a_{ij} \geq 0$ corresponds to the length of the intersection of the i-th ray with the j-th pixel. If ray i and pixel j do not intersect then $a_{ij} = 0$, see Figure 5. Stacking all equations for all the rays together leads to the linear equations $Ax = b$, and the measurements are described such that $A = (A_{\theta_1}^T \;\; A_{\theta_2}^T \;\; \dots \;\; A_{\theta_{\mathtt{nA}}}^T)^T$ and each block matrix A_{θ_i} corresponds to a different projecting angle.*

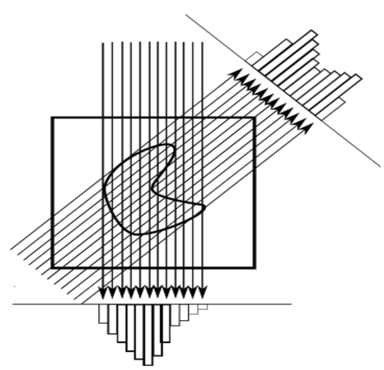

Figure 5: Parallel beam geometry set-up: a set of parallel rays is shot through the object from different directions. These are typically coined as one projection. Two projections are illustrated above. Illustration of a single projection corresponding to a measurement along one ray. A single projection corresponds to the line integral over a piecewise constant function.

In our experiments we use the MATLAB routine `paralleltomo.m` *from the AIR Tools package [28] that implements such a tomographic matrix for a given vector of angles. The Lenna grey-scaled image size is* $N = 128$ *(this means* 128×128 *pixels) and we choose the number of parallel beams to be* $nA = 100$ *for each angle, another parameter is* $p = \mathrm{round}(\sqrt{2} * 128) = 169$*. So, with this choices of parameters, we get the over-determined matrix* A *of size* $(nA * p) \times (N^2) = 18100 \times 16384$*. In this case, the data is the rows of* A *and the corresponding entries of* b*. We then divide the system* $Ax = b$ *into* 10 *sub-systems* $A_j x = b^j$*, of size* 1810×16384 *for* $j = 1, \cdots, 10$*. The time of arrival for each block is fixed and set to be 1 minute. The stopping criterion for all the algorithms is* $\|x^{k+1} - x^k\| \leq 10^{-3}$ *and the initial starting point is* $x^0 = 0$*.*

In all algorithms, besides Cimmino's method, we choose the relaxation parameters $\lambda_k \equiv 1$ *and in Cimmino's method* $\lambda_k \equiv 1.9$*. In Figure 6 we present the original Lenna image used for our recovery. In Figure 7, a run-time (in seconds) comparison between the online-block Cimmino method and the regular Cimmino method over the all data is presented. Later in Figures 8–11 we present the graphs comparing the run-times as well as the recovered images corresponding to the online-block algorithms and their regular variant operating on the all data, starting when it arrives. The term* ***Error*** *in the graphs denotes* $\|Ax^k - b\|_2$*. It can be seen in all experiments that while the difference in the recovered image is barely noticed to a naked eye, the graphs show that it is always better to apply the online-block version to obtain the needed approximation, sometimes even before the all data is available.*

Figure 6: Original image (Lenna).

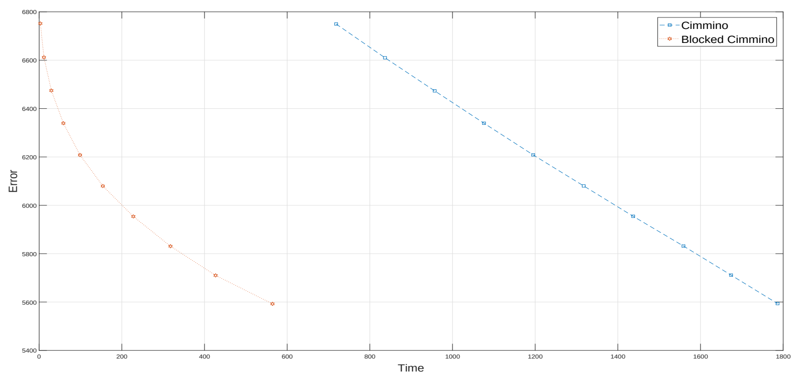

Figure 7: Cimmino method with 10 sweeps and 10 blocks.

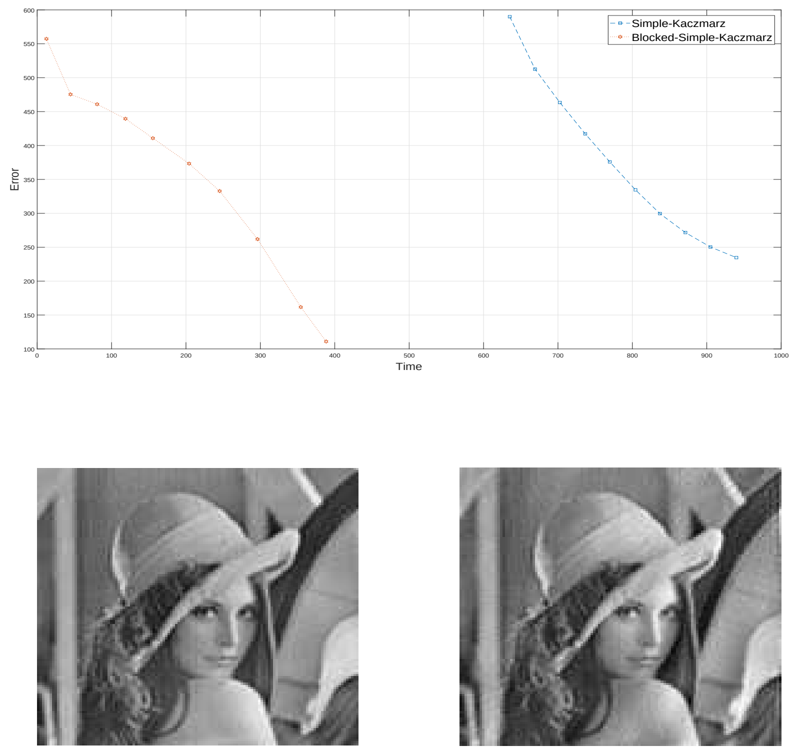

Figure 8: Comparison of Kaczmarz method

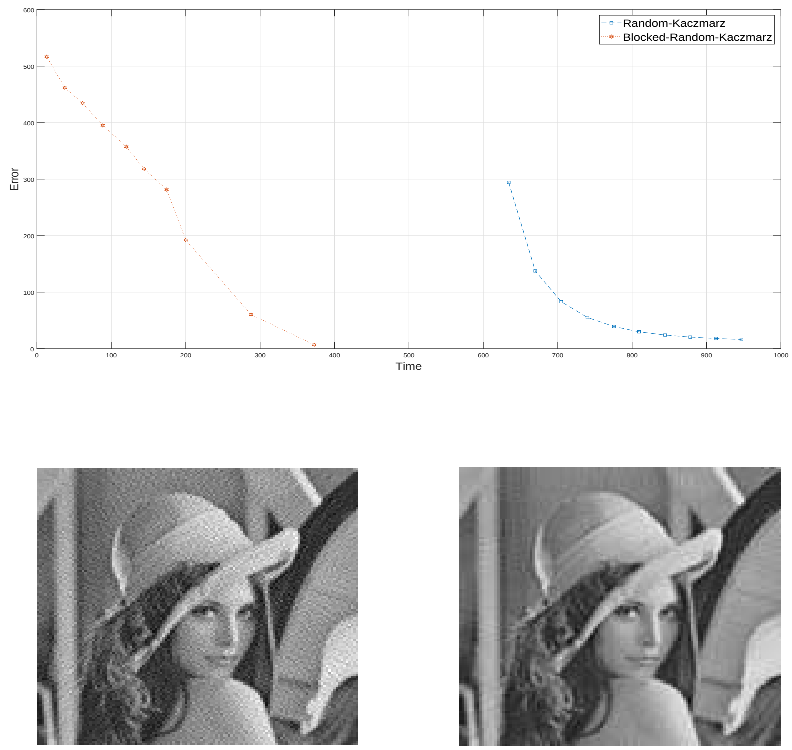

Figure 9: Comparison of randomized Kaczmarz method

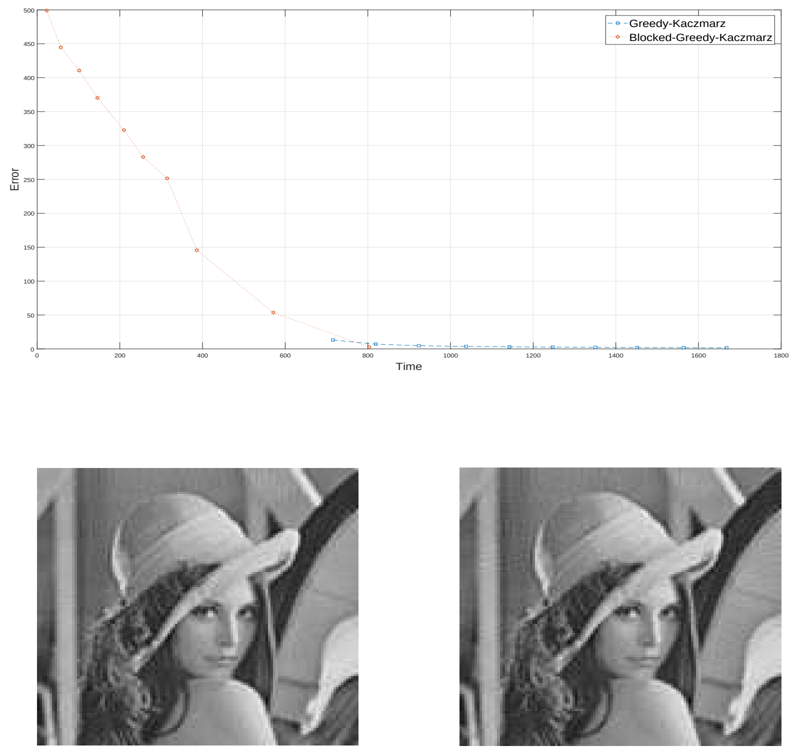

Figure 10: Comparison of greedy method

Example 4.2 ***Quadratic CFP:*** *Here we generate 10 quadratic feasibility problems in* $\mathbb{R}^{1000}$*, meaning that each set is a ball. In each experiments we increase the number of the sets and compare the performances of all the online-block algorithms and their regular variants waiting for the data to arrive. Each ball is created by picking a center* $c^i \in \mathbb{R}^{1000}$ *with coordinates randomly uniformly generated in the range* $[-5, 5]$*. Then a radius* $r_i := \|c^i\| + \alpha_i$ *was defined by adding to the center's distance from the origin* $\|c^i\|$ *a random number uniformly picked from the range* $[0, 0.1]$ *guaranteeing that the ball includes the origin, thus, yielding a consistent CFP. Initialization vectors* x^0 *were generated by randomly picking their coordinates from the range* $[-10, 10]$*. The number of constraints (balls) varied from* 200 *to* 20000*. As in Example 4.1, for each CFP, we divide the number of constraints (balls)* *into 10 blocks and determine the stopping rule:* $\|x^{k+1} - x^k\| \leq \varepsilon = 10^{-7}$. *The relaxation parameters* λ_k *is equal to 1 and in Cimmino method it was* 1.9. *In these experiments we see that as the number of constraints increases there is a big difference between the performances of the online-block schemes and their regular variants. Again this emphasize the potential applicability of these methods to online problems.*

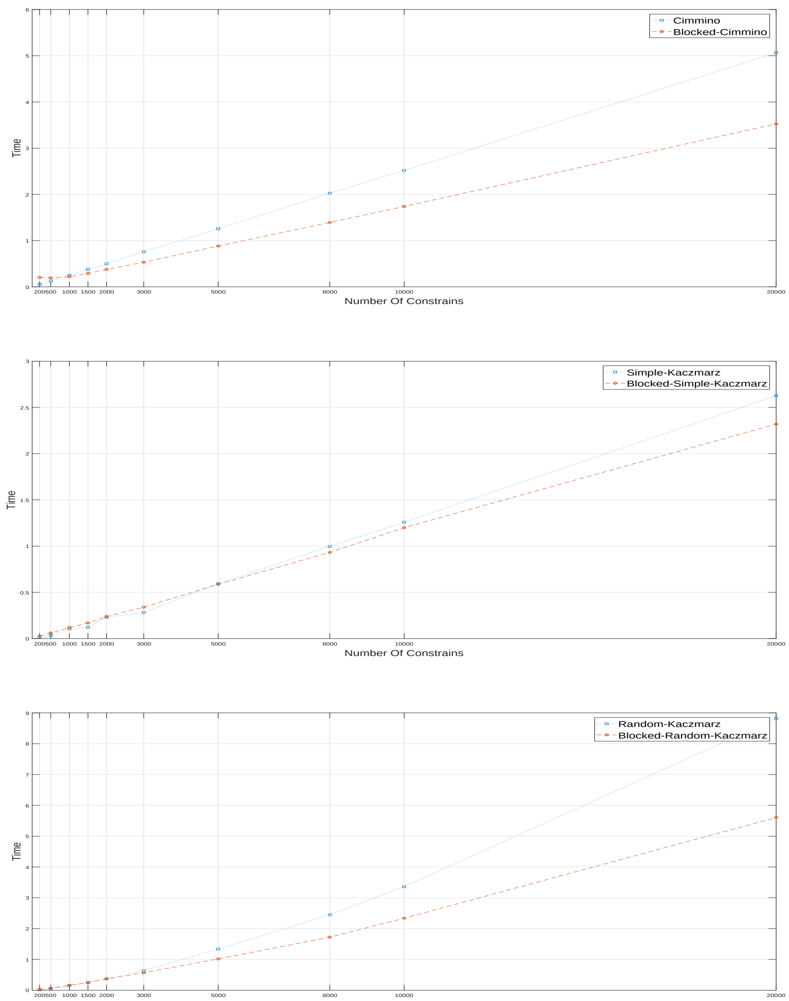

Figure 11: Run-times in seconds over 10 sweeps and 10 blocks accordingly, for Cimmino, cyclic (Kaczmarz type) and random methods.

5 Conclusions

This present project concerns with the *Common Fixed Point Problem* (CFPP) and the *Convex Feasibility Problem* (CFP) in real Hilbert spaces. We examine the situation in which the entire input, operators/sets is not available from the beginning, but provided piece-by-piece, in a sequential way.

Our motivating is the work of Ordoñez et al. [35] which presented two real-time projection methods (DROP) [1] and (CARP) [25] for solving huge, sparse and overestimated system of linear equations $Ax = b$ where $A \in \mathbb{R}^{m\times n}$ with $m \sim 10^3$ and $n \sim 10^9$, arising in the area of proton computed tomography (pCT). We present an online block-iterative scheme which is capable of operating on any block of data (operators/sets), for any finite number of iterations, before moving to the next block. The convergence proof of our scheme is based on the recent result of Reich and Zalas [37], the *Modular String Averaging* (MSA) procedure. We provide numerical experiments which show that this online block-iterative scheme produces solutions faster compared to the case when all the data is given in advance. While Ordoñez et al. [35] is focus on system of linear equations and no mathematical theory is missing, we focus on a more general framework of common fixed point problem with theoretical validity.

Although the structures of CARP and DROP does not included in the algorithmic structure T_k in Algorithm 3.2, we plan to investigate in this direction and moreover, obtain error bounds and convergence rates of this new scheme.

I would like to thank my mentor Dr. Aviv Gibali for all the support and help during the project, thank you very much.
The project was challenging and very interesting to work on, I feel that the project contributed greatly to my development as a mathematician and influenced greatly about the decision on my studies.
I was exposed to a very interesting and practical subject and learned a lot of new things.

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