אורט בראודה
המכללה האקדמית להנדסה
ORT Braude College

המחלקה למתמטיקה
Department of Mathematics

פרויקט מסכם לתואר בוגר במדעים (B.Sc)
במתמטיקה שימושית

בעיית מזעור עלויות למודל "input – output" סטוכסטי

לירון שטרוה

Cost minimization problem for stochastic input-output model

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1. Introduction

Basic Input-output model:

The input – output model is trying to solve the question:

"What level of output should each of the n industries in an economy produce, in order to satisfy the total demand for that product?"

Professor Wassily Leontief developed the input – output model in the 1930's. The model is a method to analyze the relationships between different sectors in the economy. In order to produce a product each sector may need to use its own products or other sectors products. With the input-output model one can find the outputs products number of a sector in order to satisfy the total demand.
On October 18 in 1973, Wassily Leontief won Nobel Prize in economy for his work in this area.

Input-output model assumption:

1. Each sector produces one product.
2. Each sector uses a fixed input ratio for the production of its product.

Using of Linear Algebra for the Model:

We will use basic Linear Algebra to describe and solve the model.
We will assume that there are n independent sectors: $S_1,S_2,\ldots,S_n$
Let m_{ij} be the number of units produced by sector S_i to produce one unit of sector S_j.
Let p_i be number of units produced by sector S_i.
Then the value $m_{ij}p_j$ is the number of units produced by sector S_i and consumed by sector S_j.
Let d_i be the demand for product i outside the industry.
Assuming that the production of each sector is completely consumed , the total number of units produced by sector i will be :

$$p_i = m_{1i}p_1 + m_{2i}p_2 + \cdots + m_{ii}p_i + \cdots + m_{ni}p_n + d_i, i = 1 \ldots . n \qquad (1)$$

The equations for all n sectors are:

$$\begin{cases} p_1 = m_{11}p_1 + m_{21}p_2 + \cdots + m_{n1}p_n + d_1 \\ p_2 = m_{12}p_1 + m_{22}p_2 + \cdots + m_{n2}p_n + d_2 \\ \vdots \\ p_i = m_{1i}p_1 + m_{2i}p_2 + \cdots + m_{ii}p_i + \cdots + m_{ni}p_n + d_i \\ \vdots \\ p_i = m_{1n}p_1 + m_{2n}p_2 + \cdots + m_{nn}p_n + d_n \end{cases} \qquad (2)$$

or in a matrix form:

$$p = Cp + d \qquad (3)$$

where:

$$C = \begin{pmatrix} m_{11} & m_{21} & \cdots & m_{n1} \\ m_{12} & m_{22} & \cdots & m_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ m_{1n} & m_{2n} & \cdots & m_{nn} \end{pmatrix} p = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} d = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} \qquad (4)$$

Matrix C is called input-output matrix or consumption matrix. The consumption matrix C shows the quantity of inputs needed to produce one unit of a good. The rows of the matrix represent the producing sector of the economy. The columns of the matrix represent the consuming sector of the economy. The entry m_{ij} in consumption matrix represent what percent of the total production value of sector j is spent on products from sector i. Demand vector d represents demand from the non-producing sector of the economy. Vector p represents the total amount of the product produced.

Solution of the equation:

$$p = Cp + d \rightarrow p - Cp = d \rightarrow (I - C)p = d \rightarrow p = (I - C)^{-1}d \qquad (5)$$

Assumptions:

- Consumption matrix C and demand vector d have nonnegative entries
- The inverse of the matrix (I-C) exists
- The production vector p has nonnegative entries and has unique solution for the model

Definition: Consumption matrix C can be called economically feasible if the inverse of the matrix (I-C) exists.

Example:

Let's consider an economy with two products, A and B. The demand for them is $d = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$ in order to create one A unit we need 0.2 units of B and to create one B unit we need 0.4 units of A. The Consumption matrix is:

$$C = \begin{pmatrix} 0 & 0.2 \\ 0.4 & 0 \end{pmatrix}$$

The solution can be found with equation (4) :

$$(I - C) = \begin{pmatrix} 1 & -0.2 \\ -0.4 & 1 \end{pmatrix} \rightarrow (I - C)^{-1} = \begin{pmatrix} 1.087 & 0.2174 \\ 0.4348 & 1.0870 \end{pmatrix} \rightarrow p = (I - C)^{-1}d = \begin{pmatrix} 1.087 & 0.2174 \\ 0.4348 & 1.0870 \end{pmatrix} \begin{pmatrix} 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 8.4783 \\ 7.3913 \end{pmatrix}$$

In this case the inverse matrix $(I - C)^{-1}$ exists, so Matrix C is economically feasible.

Minimum problem with input – output model

In real life section owners will want not only to solve the input-output problem, but also desire to minimize the costs function and to maximize profits.

Let's assume that for each product the cost to produce is c_i and the price in the market it P_i. Then the profit maximization problem will be:

$$\max[\sum_{i=1}^{n}(P_i - c_i)p_i]$$

If we will assume that the price in the market for each product is fixed then we can solve the production cost minimization problem instead:

$$min[\sum_{i=1}^{n} c_i p_i] \quad (6)$$

2. Problem Statement

2.1 Stochastic Input Output model

The objective of this project is studying the stochastic version if the input-output model.

The main idea will be the same, but instead of known / fixed coefficients on matrix C. we will assume they are stochastic variables with a known distribution.

We will assume that there are n independent sectors: $S_1, S_2, \ldots, S_n$
Let m_{ij} be the number of produced units by sector S_i to produce one unit of sector S_j , but now m_{ij} is a stochastic variable with a known distribution.
Let p_i be the number of units produced by sector S_i.
Then the value $m_{ij}p_j$ is the number of units produced by sector S_i and consumed by sector S_j.
Let d_i be the demand for product i outside the industry.
Then the total number of units produced by sector i are:

$$p_i = m_{1i}p_1 + m_{2i}p_2 + \cdots + m_{ii}p_i + \cdots + m_{ni}p_n + d_i, i = 1 \ldots . n \quad (7)$$

The equation for all n sectors will be:

$$\begin{cases} p_1 = m_{11}p_1 + m_{21}p_2 + \cdots + m_{n1}p_n + d_1 \\ p_2 = m_{12}p_1 + m_{22}p_2 + \cdots + m_{n2}p_n + d_2 \\ \vdots \\ p_i = m_{1i}p_1 + m_{2i}p_2 + \cdots + m_{ii}p_i + \cdots + m_{ni}p_n + d_i \\ \vdots \\ p_i = m_{1n}p_1 + m_{2n}p_2 + \cdots + m_{nn}p_n + d_n \end{cases} \quad (8)$$

Or in a matrix form:

$$p = Cp + d \quad (9)$$

Where:

$$C = \begin{pmatrix} m_{11} & m_{21} & \cdots & m_{n1} \\ m_{12} & m_{22} & \cdots & m_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ m_{1n} & m_{2n} & \cdots & m_{nn} \end{pmatrix} p = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} d = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} \quad (10)$$

Now we can’t calculate the inverse matrix to find the solution, instead we can search for a solution which will solve the equation with a probability above a specific number.

In this paper we will describe and solve the Stochastic Input Output in a specific case.

2.2 Minimum problem with stochastic input – output model

In the stochastic input – output model the costs to produce one unit are not known or fixed. They are stochastic variables with a known distribution.

The costs can be represented as a vector:

$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$, c_i is the cost for producing one unit of p_i.

Now the problem is formatted as follows:

$$\min c^T p \quad (11)$$

subject to:

$$P(p - Cp \geq d) \geq 1 - \alpha \quad (12)$$

where α is a probability, $0 < \alpha < 1$

3 Solution of minimization problem for stochastic input-output model

In this project we study the following particular case of the problem.

Our assumptions are:

$$n = 2$$

$$m_{11} = 0, \mathrm{m}_{12} \sim \mathrm{U}[0, \mathrm{b}], \mathrm{m}_{22} = 0, \mathrm{m}_{21} \sim \mathrm{U}[0, \mathrm{d}]$$

Where $X \sim U[x_1, x_2]$ means that X is a uniform distributed random variable on $[x_1, x_2]$

The assumptions $\mathrm{m}_{11} = 0, \mathrm{m}_{22} = 0$ mean that for the production of product p_i there is no need for the same product.

In such case the matrices are:

$$C = \begin{pmatrix} 0 & m_{12} \\ m_{21} & 0 \end{pmatrix} \; p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \; d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \quad (13)$$

The corresponding equations are:

$$\begin{cases} p_1 = m_{12} p_2 + d_1 \\ p_2 = m_{21} p_1 + d_2 \end{cases} \rightarrow \begin{cases} p_1 - m_{12} p_2 = d_1 \\ p_2 - m_{21} p_1 = d_2 \end{cases} \quad (14)$$

Since m_{12} , m_{21} are stochastic variables we can't solve the problem directly , but we can search for p_1 , p_2 which will solve the problem with specific probability,

$$P(p_1 - m_{12} p_2 \geq d_1, p_2 - m_{21} p_1 \geq d_2) \geq 1 - \alpha \quad , \quad (15)$$

where α is a small probability $0 \le \alpha \le 1$

3.1 Calculation of probability functions

The probability function for uniform stochastic variable $x \sim U[0, b]$ is:

$$P(x \le y) = F(y) = \begin{cases} 0, y < 0 \\ \frac{y}{b}, 0 \le y \le b \\ 1, y > b \end{cases}$$

Therefore:

$$P(p_1 - m_{12}p_2 \ge d_1) = P(-m_{12}p_2 \ge d_1 - p_1) = P\left(m_{12} \le -\frac{d_1 - p_1}{p_2}\right)$$

$$= \begin{cases} 0\,, -\frac{d_1-p_1}{p_2} < 0 \\ \frac{-\frac{d_1-p_1}{p_2}}{b}, 0 \le -\frac{d_1-p_1}{p_2} \le b \\ 1, -\frac{d_1-p_1}{p_2} > b \end{cases} = \begin{cases} 0\,, -\frac{d_1-p_1}{p_2} < 0 \\ \frac{-d_1+p_1}{p_2 b}, 0 \le -\frac{d_1-p_1}{p_2} \le b \\ 1, -\frac{d_1-p_1}{p_2} > b \end{cases} \quad (16)$$

Similarly:

$$P(p_2 - m_{21}p_1 \ge d_2) = P(-m_{21}p_1 \ge d_2 - p_2) = P\left(m_{21} \le -\frac{d_2 - p_2}{p_1}\right)$$

$$= \begin{cases} 0\,, -\frac{d_2-p_2}{p_1} < 0 \\ \frac{-\frac{d_2-p_2}{p_1}}{d}, 0 \le -\frac{d_2-p_2}{p_1} \le d \\ 1, -\frac{d_2-p_2}{p_1} > d \end{cases} = \begin{cases} 0\,, -\frac{d_2-p_2}{p_1} < 0 \\ \frac{-d_2+p_2}{p_1 d}, 0 \le -\frac{d_2-p_2}{p_1} \le d \\ 1, -\frac{d_2-p_2}{p_1} > d \end{cases} \quad (17)$$

Now we will calculate $P(p_1 - m_{12}p_2 \ge d_1)\ P(p_2 - m_{21}p_1 \ge d_2)$

$$P(p_1 - m_{12}p_2 \ge d_1)P(p_2 - m_{21}p_1 \ge d_2) =$$

$$\begin{cases} 0 & , -\frac{d_1 - p_1}{p_2} < 0\ or\ -\frac{d_2 - p_2}{p_1} < 0 \\ \frac{-d_1 + p_1}{p_2 b} \cdot \frac{-d_2 + p_2}{p_1 d} & , 0 \le -\frac{d_1 - p_1}{p_2} \le b\ and\ 0 \le -\frac{d_2 - p_2}{p_1} \le d \\ \frac{-d_1 + p_1}{p_2 b} & , 0 \le -\frac{d_1 - p_1}{p_2} \le b\ and\ -\frac{d_2 - p_2}{p_1} > d \\ \frac{-d_2 + p_2}{p_1 d} & , -\frac{d_1 - p_1}{p_2} > b\ and\ 0 \le -\frac{d_1 - p_2}{p_1} \le d \\ 1 & , -\frac{d_1 - p_1}{p_2} > b\ and\ -\frac{d_2 - p_2}{p_1} > d \end{cases} \quad (18)$$

Simplification of the probability function:

Case 1:

$$P(p_1 - m_{12}p_2 \geq d_1)P(p_2 - m_{21}p_1 \geq d_2) = 0, -\frac{d_1 - p_1}{p_2} < 0 \; or \; -\frac{d_2 - p_2}{p_1} < 0$$

$$-\frac{d_1 - p_1}{p_2} < 0 \to -d_1 + p_1 < 0 \to p_1 < d_1$$

For the same reasons the second condition becomes:

$$-\frac{d_2 - p_2}{p_1} < 0 \to p_2 < d_2$$

In conclusion the conditions to get $P(p_1 - m_{12}p_2 \geq d_1)P(p_2 - m_{21}p_1 \geq d_2) = 0$ are:

$$p_1 < d_2 \; or \; p_2 < d_1 \qquad (19)$$

In this case the economy will not reach the required probability .This case is less interesting, so we will remove it from our studying.

Case 2:

$$P(p_1 - m_{12}p_2 \geq d_1)P(p_2 - m_{21}p_1 \geq d_2) = \frac{-d_1 + p_1}{p_2 b} \cdot \frac{-d_2 + p_2}{p_1 d},$$

$$a \leq -\frac{d_1 - p_1}{p_2} \leq b \; and \; c \leq -\frac{d_1 - p_2}{p_1} \leq d$$

$$0 \leq -\frac{d_1 - p_1}{p_2} \leq b \to 0 \leq -d_1 + p_1 \leq bp_2 \to d_1 \leq p_1 \leq bp_2 + d_1$$

For the same reasons the second condition becomes:

$$d_2 \leq p_2 \leq dp_1 + d_2$$

$$d_1 \leq p_1 \leq bp_2 + d_1 \; and \; d_2 \leq p_2 \leq dp_1 + d_2 \qquad (20)$$

Case 3 :

$$P(p_1 - m_{12}p_2 \geq d_1)P(p_2 - m_{21}p_1 \geq d_2) = \frac{-d_1 + p_2}{p_2 b},$$

$$0 \leq -\frac{d_1 - p_1}{p_2} \leq b \; and \; -\frac{d_2 - p_2}{p_1} > d$$

From case 2 the first condition becomes:

$$0 \leq -\frac{d_1 - p_1}{p_2} \leq b \rightarrow d_1 \leq p_1 \leq bp_2 + d_1$$

The second condition:

$$-\frac{d_2 - p_2}{p_1} > d \rightarrow p_2 - d_2 > dp_1 \rightarrow p_2 > dp_1 + d_2$$

$$d_1 \leq p_1 \leq bp_2 + d_1 \ and \ p_2 > dp_1 + d_2 \quad (21)$$

Case 4:

$$P(p_1 - m_{12}p_2 \geq d_1)P(p_2 - m_{21}p_1 \geq d_2) = \frac{-d_1 + p_2}{p_1 d},$$

$$-\frac{d_1 - p_1}{p_2} > b \ \ and \ \ 0 \leq -\frac{d_1 - p_2}{p_1} \leq d$$

$$-\frac{d_1 - p_1}{p_2} > b \rightarrow -d_1 + p_1 > bp_2 \rightarrow p_1 > bp_2 + d_1$$

From case 2, the second condition becomes:

$$d_2 \leq p_2 \leq dp_1 + d_2$$

$$p_1 > bp_2 + d_1 \ and \ d_2 \leq p_2 \leq dp_1 + d_2 \quad (22)$$

Case 5:

$$P(p_1 - m_{12}p_2 \geq d_1)P(p_2 - m_{21}p_1 \geq d_2) = 1, -\frac{d_1 - p_1}{p_2} > b \ and \ -\frac{d_2 - p_2}{p_1} > d$$

From cases 3, 4 the conditions for the case become:

$$p_1 > bp_2 + d_1 \ and \ p_2 > dp_1 + d_2 \quad (23)$$

After rewriting the conditions by using (19)-(23) the probability function is:

$$P(p_1 - m_{12}p_2 \geq d_1)P(p_2 - m_{21}p_1 \geq d_2) =$$

$$\begin{cases} 0 & ,p_1 < d_1 \; or \; p_2 < d_2 \\ \frac{-d_1+p_1}{p_2 b} \cdot \frac{-d_2+p_2}{p_1 d} & ,d_1 \le p_1 \le bp_2 + d_1 \; and \; d_2 \le p_2 \le dp_1 + d_2 \\ \frac{-d_1+p_1}{p_2 b} & ,d_1 \le p_1 \le bp_2 + d_1 \; and \; p_2 > dp_1 + d_2 \\ \frac{-d_2+p_2}{p_1 d} & ,p_1 > bp_2 + d_1 \; and \; d_2 \le p_2 \le dp_1 + d_2 \\ 1 & ,p_1 > bp_2 + d_1 \; and \; p_2 > dp_1 + d_2 \end{cases} \tag{24}$$

Looking again on condition (23):

$p_1 > bp_2 + d_1 \; and \; p_2 > dp_1 + d_2$, if $p_2 = dp_1 + d_2$ then:

$$p_1 > bp_2 + d_1 \rightarrow p_1 > b(dp_1 + d_2) + d_1 \rightarrow p_1 > bdp_1 + bd_2 + d_1$$

Therefore if $bd \ge 1$then the Non-equality will never be satisfied. In this project we will assume that $bd \ge 1$ because the case which

$$P(p_1 - m_{12}p_2 \ge d_1)P(p_2 - m_{21}p_1 \ge d_2) = 1 \tag{25}$$

is less interesting, eq. (20) means that the economy meets the demand and production on 100% of the cases.

After rewriting the conditions and under the assumption:

$$bd \ge 1 \tag{26}$$

the probability function becomes:

$$P(p_1 - m_{12}p_2 \ge d_1)P(p_2 - m_{21}p_1 \ge d_2) =$$

$$\begin{cases} \frac{-d_1+p_1}{p_2 b} \cdot \frac{-d_2+p_2}{p_1 d} & ,d_1 \le p_1 \le bp_2 + d_1 \; and \; d_2 \le p_2 \le dp_1 + d_2 \quad (27.1) \\ \frac{-d_1+p_1}{p_2 b} & ,d_1 \le p_1 \le bp_2 + d_1 \; and \; p_2 > dp_1 + d_2 \quad (27.2) \\ \frac{-d_2+p_2}{p_1 d} & ,p_1 > bp_2 + d_1 \; and \; d_2 \le p_2 \le dp_1 + d_2 \quad (27.3) \end{cases}$$

<u>Experimental testing (experimental_testing_1_a.m):</u>

The program randomly chooses m_{12} , m_{21} , when m_{12} ~ U[0,b] , m_{21} ~[0,d] . The other parameters are fixed: $b = 1, d = 1.01, d_1 = 1, d_2 = 1$ and p_1, p_2 are chosen for every one of the cases. Then program runs 100000000 times for every case and sum the number of runs where for ransoms variables m_{12}, m_{21} the following conditions are satisfied:

$$\begin{cases} p_1 - m_{12}p_2 \ge d_1 \\ p_2 - m_{21}p_1 \ge d_2 \end{cases}$$

Then it compares it to the probability function (24).

Results:

Id	p_1	p_2	Case	P based on (24)	P based on testing
1	0.5	0.9	(24.1)	0	0
2	1.5	1.9	(24.2)	0.156331	0.156339
3	1.5	5	(24.3)	0.1	0.10004
4	5	1.5	(24.4)	0.099	0.09899

Lemma 1: If $1 - bd(1-\alpha) = 0$ then the probability function in case 1 receive only negative values.

Proof:

The probability function and the conditions in case 1:

$$P(p_1 - m_{12}p_2 \geq d_1)P(p_2 - m_{21}p_1 \geq d_2) = \frac{-d_1 + p_2}{p_2 b} \cdot \frac{-d_1 + p_2}{p_1 d}, d_1 \leq p_1 \leq bp_2 + d_1 \; and \; d_2 \leq p_2 \leq dp_1 + d_2$$

We will demand that the probability function will satisfy the following Inequality:

$$P(p_1 - m_{12}p_2 \geq d_1)P(p_2 - m_{21}p_1 \geq d_2) = \frac{-d_1 + p_2}{p_2 b} \cdot \frac{-d_1 + p_2}{p_1 d} \geq 1 - \alpha$$

where $0 \leq \alpha \leq 1$.

$$\frac{-d_1 + p_1}{p_2 b} \cdot \frac{-d_2 + p_2}{p_1 d} \geq 1 - \alpha \rightarrow (-d_1 + p_1)(-d_2 + p_2) \geq bdp_1 p_2 (1-\alpha) \rightarrow$$

$$d_1 d_2 - d_1 p_2 - d_2 p_1 + p_1 p_2 - bdp_1 p_2 (1-\alpha) \geq 0 \rightarrow$$

$$d_1 d_2 - d_1 p_2 - d_2 p_1 + p_1 p_2 [1 - bd(1-\alpha)] \geq 0$$

Let $f(p) = d_1 d_2 - d_1 p_2 - d_2 p_1 + p_1 p_2 [1 - bd(1-\alpha)]$ (28)

$$1 - bd(1-\alpha) = 0 \rightarrow f(p) = d_1 d_2 - d_1 p_2 - d_2 p_1$$

The conditions for case 1 are:

$$d_1 \leq p_1 \leq bp_2 + d_1 \; and \; d_2 \leq p_2 \leq dp_1 + d_2$$

Therefore:

$$f(p) = d_1 d_2 - d_1 p_2 - d_2 p_1 \leq d_1 d_2 - d_1 d_2 - d_1 d_2 = -d_1 d_2 < 0$$

So for every p_1,p_2 if $1 - bd(1-\alpha) = 0$ the probability function doesn't satisfy the condition $f(p) \geq 0$ ■

Lemma 2: if $1 - bd(1-\alpha) > 0$ then the probability function in case 1 can receive positive values.

Proof:

$$1 - bd(1-\alpha) > 0$$

$$1 - bd(1-\alpha) > 0 \rightarrow 1 > bd(1-\alpha) > 0$$

The probability function in this case is:

$$f(p) = d_1 d_2 - d_1 p_2 - d_2 p_1 + p_1 p_2 [1 - bd(1-\alpha)]$$

And the conditions are:

$$d_1 \le p_1 \le bp_2 + d_1 \; and \; d_2 \le p_2 \le dp_1 + d_2$$

Let $\varepsilon = [1 - bd(1-\alpha)] > 0$ .Then:

$$f(p) = d_1 d_2 - d_1 p_2 - d_2 p_1 + p_1 p_2 \varepsilon$$
$$\ge d_1 d_2 - d_1(dp_1 + d_2) - d_2(bp_2 + d_1) + d_1 d_2 \varepsilon =$$

$$= d_1 d_2 - d_1 dp_1 - d_1 d_2 - d_2 bp_2 - d_1 d_2 + d_1 d_2 \varepsilon =$$

$$= -d_1 dp_1 - d_2 bp_2 + d_1 d_2(\varepsilon - 1)$$

We demand that

$f(p) \ge 0 \rightarrow -d_1 dp_1 - d_2 bp_2 + d_1 d_2(\varepsilon - 1) \ge 0$, but $-d_1 dp_1 - d_2 bp_2 + d_1 d_2(\varepsilon - 1)$ is a negative number.

$$f(p) = d_1 d_2 - d_1 p_2 - d_2 p_1 + p_1 p_2 \varepsilon$$
$$\le d_1 d_2 - d_1 d_2 - d_1 d_2 + (bp_2 + d_1)(dp_1 + d_2)\varepsilon$$

$$= -d_1 d_2 + (bp_2 dp_1 + bp_2 d_2 + d_1 dp_1 + d_1 d_2)\varepsilon$$
$$= d_1 d_2(\varepsilon - 1) + (bp_2 dp_1 + bp_2 d_2 + d_1 dp_1)\varepsilon$$

We found upper and lower bands for $f(p)$:

$$d_1 d_2(\varepsilon - 1) + (bp_2 dp_1 + bp_2 d_2 + d_1 dp_1)\varepsilon \ge f(p)$$
$$\ge -d_1 dp_1 - d_2 bp_2 + d_1 d_2(\varepsilon - 1)$$

Therefore if $d_1 d_2(\varepsilon - 1) + (bp_2 dp_1 + bp_2 d_2 + d_1 dp_1)\varepsilon \ge 0$ then $f(p)$ might be positive.

$$d_1 d_2(\varepsilon - 1) + (bp_2 dp_1 + bp_2 d_2 + d_1 dp_1)\varepsilon \ge 0 \rightarrow$$

$$p_1 \varepsilon(bp_2 d + d_1 d) \ge d_1 d_2(1 - \varepsilon) - bp_2 d_2 \varepsilon \rightarrow$$

$$p_1 \ge \frac{d_1 d_2(1-\varepsilon) - bp_2 d_2 \varepsilon}{\varepsilon(bp_2 d + d_1 d)} \qquad (29)$$

We can't calculate when $f(p) \geq 0$ because $f(p) \geq -d_1dp_1 - d_2bp_2 + d_1d_2(\varepsilon - 1)$ and $\geq -d_1dp_1 - d_2bp_2 + d_1d_2(\varepsilon - 1)$ is a negative number, but we can say that when

$$p_1 < \frac{d_1d_2(1-\varepsilon) - bp_2d_2\varepsilon}{\varepsilon(bp_2d + d_1d)}$$

then :

$$f(p) < 0$$

Now let us show that $d_1 > \frac{d_1d_2(1-\varepsilon)-bp_2d_2\varepsilon}{\varepsilon(bp_2d+d_1d)}$ therefore the condition for p_1 stay $p_1 \geq d_1$:

$$d_1 - \frac{d_1d_2(1-\varepsilon) - bp_2d_2\varepsilon}{\varepsilon(bp_2d + d_1d)} = d_1 - \frac{d_1d_2(1-\varepsilon) - bp_2d_2\varepsilon}{\varepsilon d(bp_2 + d_1)}$$
$$\geq d_1 - \frac{d_1d_2(1-\varepsilon) - bp_2d_2\varepsilon}{\varepsilon dp_1} = \frac{d_1\varepsilon dp_1 - d_1d_2(1-\varepsilon) + bp_2d_2\varepsilon}{\varepsilon dp_1}$$

$$d_1\varepsilon dp_1 - d_1d_2(1-\varepsilon) + bp_2d_2\varepsilon = d_1\varepsilon dp_1 - d_1d_2 + d_1d_2\varepsilon + bp_2d_2\varepsilon =$$

$$d_1dp_1[1 - bd(1-\alpha)] - d_1d_2 + d_1d_2[1 - bd(1-\alpha)] + bp_2d_2[1 - bd(1-\alpha)] =$$

$d_1dp_1[1 - bd(1-\alpha)] - d_1d_2 + d_1d_2 - d_1d_2bd(1-\alpha) + bp_2d_2[1 - bd(1-\alpha)] = d_1dp_1[1 - bd(1-\alpha)] - d_1d_2bd(1-\alpha) + bp_2d_2[1 - bd(1-\alpha)] \geq d_1dp_1[1 - bd(1-\alpha)] + bp_2d_2[1 - bd(1-\alpha)] \geq 0$ ■

Lemma 3: if $1 - bd(1-\alpha) < 0$ then the probability function in case 1 receive only negative values.

Proof:

$$1 - bd(1-\alpha) < 0$$

Let $-\varepsilon = [1 - bd(1-\alpha)]$, $\varepsilon > 0$ then the probability function in this case is:

$$f(p) = d_1d_2 - d_1p_2 - d_2p_1 - p_1p_2\varepsilon$$

And the conditions are:

$$d_1 \leq p_1 \leq bp_2 + d_1 \; and \; d_2 \leq p_2 \leq dp_1 + d_2$$

Therefore:

$$f(p) = d_1d_2 - d_1p_2 - d_2p_1 - p_1p_2\varepsilon \leq d_1d_2 - d_1d_2 - d_1d_2 - d_1d_2\varepsilon$$
$$= -d_1d_2 - d_1d_2\varepsilon < 0$$

So for every p_1,p_2 if $1 - bd(1-\alpha) < 0$ the probability function doesn't satisfy the condition $f(p) \geq 0$■

With Lemma 1, 2, 3 and other assumptions we made the conditions for a solution in case 1 of the probability function are:

$$\begin{cases} bd \geq 1 \\ 1 - bd(1-\alpha) > 0 \end{cases} \tag{30}$$

Experimental testing:

The program randomly chooses p_1. The other parameters are fixed: $b = 1, d = 1.01, d_1 = 1, d_2 = 1, p_2 = 3$. Then program runs 100000000 times for cases I,II and III and checks if the probability function is equal or higher than 0 :

Case	$1 - bd(1-\alpha)$	Number of times $f(p) \geq 0$	Number of time $f(p) < 0$
I	0	0	100000000
II	0.0910	0	100000000
II	0.1920	58267362	41732638
II	0.4950	100000000	0
III	-0.0100	0	100000000
III	-0.0090	0	100000000
III	-0.0043	0	100000000
III	-3.0400e-004	0	100000000

We can see that in cases I and III we always get negative values for $f(p)$ and in case II , we get mixed values.

3.2 Derivation of constraints set

3.2.1 Case 1

Now we would like to find the intersection of the conditions in case 1 and the probability function in case 1.

The probability function in case 1 is:

$$f(p) = d_1 d_2 - d_1 p_2 - d_2 p_1 + p_1 p_2[1 - bd(1-\alpha)]$$

$f(p)$ is a hyperbolic function. In figure 1 you can see an example of the line $f(p) = 0$.

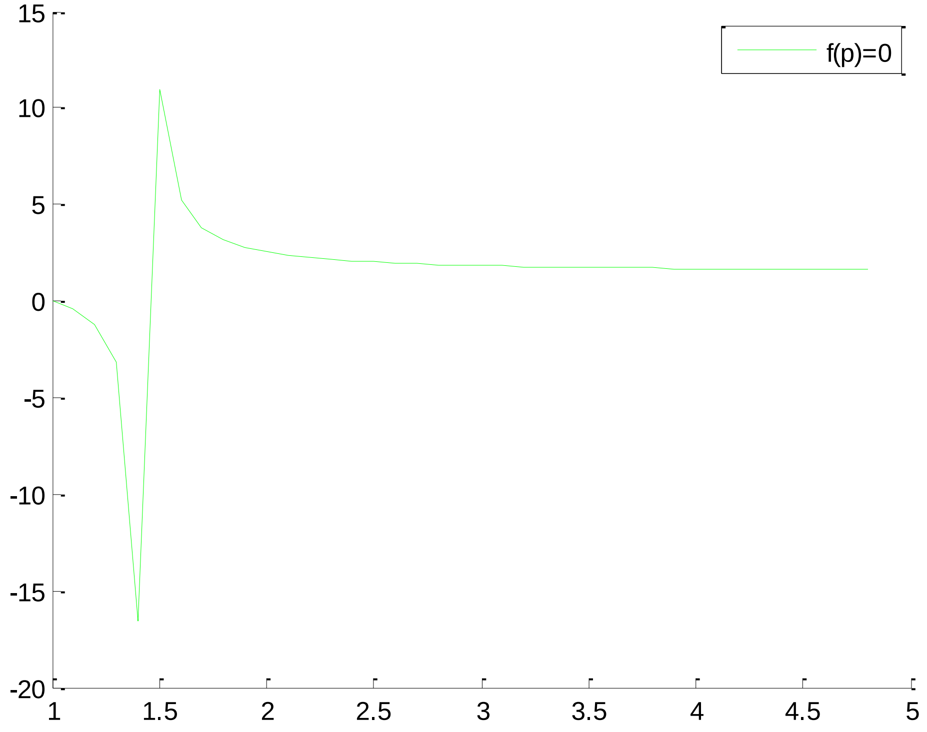

Figure 1

The conditions in case 1 are:

$$d_1 \leq p_1 \leq bp_2 + d_1 \ and \ d_2 \leq p_2 \leq dp_1 + d_2$$

Definition:

Let $S_2 = \{(p_1, p_1) | d_1 \leq p_1 \leq bp_2 + d_1 \ and \ d_2 \leq p_2 \leq dp_1 + d_2\}$

We will find the intersection of $f(p) = 0$ with the conditions defined at S_2:

$$\begin{cases} p_1 = d_1 \\ p_1 = bp_2 + d_1 \\ p_2 = d_2 \\ p_2 = dp + d_2 \end{cases}$$

In figure 2 you can see an example of the intersection of the line $f(p) = 0$ and S_2.

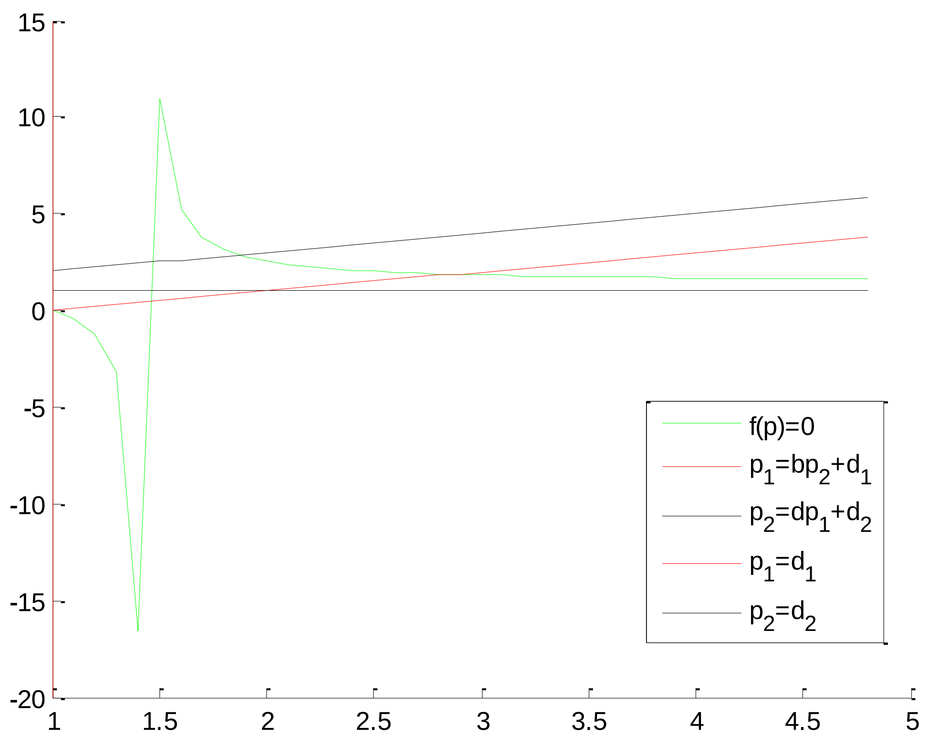

Figure 2

Lemma 4: There is no intersection point between $f(p) = 0$ and $d_1 = p_1$ which belongs to S_2.

Proof:

$$\begin{cases} f(p) = d_1 d_2 - d_1 p_2 - d_2 p_1 + p_1 p_2 [1 - bd(1-\alpha)] = 0 \\ d_1 = p_1 \end{cases}$$

$$f(p) = d_1 d_2 - d_1 p_2 - d_2 p_1 + p_1 p_2 [1 - bd(1-\alpha)]$$
$$= d_1 d_2 - d_1 p_2 - d_2 d_1 + d_1 p_2 [1 - bd(1-\alpha)] =$$

$$d_1 p_2 [1 - bd(1-\alpha) - 1] = d_1 p_2 bd(1-\alpha) = 0 \rightarrow p_2 = 0$$

In S_2 we demand $d_2 \leq p_2$ therefore $(p_1, p_2) = (d_1, 0) \notin S_2$. ■

Lemma 5: There is an intersection point between $f(p) = 0$ and $p_1 = bp_2 + d_1$ which belongs to S_2 and the point is $(p_1, p_2) = \left(bp_2 + d_1, p_2\right) = \left(b \frac{d_2 + d(1-\alpha)d_1}{[1 - bd(1-\alpha)]} + d_1, \frac{d_2 + d(1-\alpha)d_1}{[1 - bd(1-\alpha)]}\right)$.

Proof:

$$\begin{cases} f(p) = d_1 d_2 - d_1 p_2 - d_2 p_1 + p_1 p_2 [1 - bd(1-\alpha)] = 0 \\ p_1 = bp_2 + d_1 \end{cases}$$

$$f(x) = d_1 d_2 - d_1 p_2 - d_2(bp_2 + d_1) + (bp_2 + d_1)p_2[1 - bd(1-\alpha)] =$$

$$d_1 d_2 - d_1 p_2 - d_2 bp_2 - d_1 d_2 + (bp_2{}^2 + d_1 p_2)[1 - bd(1-\alpha)] =$$

$$-d_1 p_2 - d_2 bp_2 + bp_2{}^2[1 - bd(1-\alpha)] + d_1 p_2[1 - bd(1-\alpha)] =$$

$$-d_1 p_2 - d_2 bp_2 + bp_2{}^2[1 - bd(1-\alpha)] + d_1 p_2 - bd(1-\alpha)d_1 p_2 =$$

$$-d_2 bp_2 + bp_2{}^2[1 - bd(1-\alpha)] - bd(1-\alpha)d_1 p_2 =$$

$$p_2(-d_2 b + bp_2[1 - bd(1-\alpha)] - bd(1-\alpha)d_1) = 0$$

$$\rightarrow \begin{cases} p_2 = 0, \mathit{not\ in\ the\ range} \\ -d_2 b + bp_2[1 - bd(1-\alpha)] - bd(1-\alpha)d_1 = 0 \end{cases}$$

$$bp_2[1 - bd(1-\alpha)] = d_2 b + bd(1-\alpha)d_1 \rightarrow$$

$$p_2 = \frac{d_2 b + bd(1-\alpha)d_1}{b[1 - bd(1-\alpha)]} = \frac{d_2 + d(1-\alpha)d_1}{[1 - bd(1-\alpha)]}$$

The intersection point $p_2 = \frac{d_2 + d(1-\alpha)d_1}{[1-bd(1-\alpha)]}$ is in the range:

$d_2 + d(1-\alpha)d_1 > d_2$ And $1 > [1 - bd(1-\alpha)] > 0$, therefore $\frac{d_2 + d(1-\alpha)d_1}{[1-bd(1-\alpha)]} > d_2$

$$dp_1 + d_2 = d(bp_2 + d_1) + d_2 = dbp_2 + dd_1 + d_2 > p_2 \rightarrow$$

$$(p_1, p_2) = (bp_2 + d_1, p_2) = \left(b\frac{d_2 + d(1-\alpha)d_1}{[1 - bd(1-\alpha)]} + d_1, \frac{d_2 + d(1-\alpha)d_1}{[1 - bd(1-\alpha)]}\right) \in S_2 \blacksquare$$

Lemma 6: There is no intersection point between $f(p) = 0$ and $d_2 = p_2$ which belongs to S_2.

Proof:

$$\begin{cases} f(p) = d_1 d_2 - d_1 p_2 - d_2 p_1 + p_1 p_2[1 - bd(1-\alpha)] = 0 \\ d_2 = p_2 \end{cases}$$

$$f(p) = d_1 d_2 - d_1 p_2 - d_2 p_1 + p_1 p_2[1 - bd(1-\alpha)] = (p)$$
$$= d_1 d_2 - d_1 d_2 - d_2 p_1 + p_1 d_2[1 - bd(1-\alpha)]$$

$$= -d_2 p_1 + p_1 d_2[1 - bd(1-\alpha)]$$

$$= p_1(-d_2 + d_2[1 - bd(1-\alpha)]) = 0 \rightarrow p_1 = 0$$

In S_2 we demand $d_1 \leq p_1$ therefore $(p_1, p_2) = (0, d_2) \notin S_2$. $\blacksquare$

Lemma 7: There is an intersection point between $f(p) = 0$ and $p_2 = dp_1 + d_2$ which belongs to S_2 and the point is
$(p_1, p_2) = \left(\frac{d_1 + d_2[b(1-\alpha)]}{1-[bd(1-\alpha)]}, dp_1 + d_2\right) = \left(\frac{d_1 + d_2[b(1-\alpha)]}{1-[bd(1-\alpha)]}, d\frac{d_1 + d_2[b(1-\alpha)]}{1-[bd(1-\alpha)]} + d_2\right)$.

Proof:

$$\begin{cases} f(p) = d_1d_2 - d_1p_2 - d_2p_1 + p_1p_2[1 - bd(1-\alpha)] = 0 \\ p_2 = dp_1 + d_2 \end{cases}$$

$$f(p) = d_1d_2 - d_1p_2 - d_2p_1 + p_1p_2[1 - bd(1-\alpha)] =$$

$$d_1d_2 - d_1(dp_1 + d_2) - d_2p_1 + p_1(dp_1 + d_2)[1 - bd(1-\alpha)] =$$

$$d_1d_2 - d_1dp_1 - d_1d_2 - d_2p_1 + (dp_1{}^2 + d_2p_1)[1 - bd(1-\alpha)] =$$

$$-d_1dp_1 - d_2p_1 + (dp_1{}^2 + d_2p_1)[1 - bd(1-\alpha)] =$$

$$-d_1dp_1 - d_2p_1 + dp_1{}^2 + d_2p_1 - (dp_1{}^2 + d_2p_1)[bd(1-\alpha)] =$$

$$-d_1dp_1 + dp_1{}^2 - (dp_1{}^2 + d_2p_1)[bd(1-\alpha)] =$$

$$-d_1dp_1 + dp_1{}^2 - p_1(dp_1 + d_2)[bd(1-\alpha)] =$$

$$p_1(-d_1d + dp_1 - (dp_1 + d_2)[bd(1-\alpha)]) = 0$$

$$\rightarrow \begin{cases} p_1 = 0\,, not\ in\ the\ range \\ -d_1d + dp_1 - (dp_1 + d_2)[bd(1-\alpha)] = 0 \end{cases}$$

$$-d_1d + dp_1 - (dp_1 + d_2)[bd(1-\alpha)] = 0 \rightarrow$$

$$-d_1d + dp_1 - dp_1[bd(1-\alpha)] - d_2[bd(1-\alpha)] = 0 \rightarrow$$

$$dp_1 - dp_1[bd(1-\alpha)] = d_1d + d_2[bd(1-\alpha)]$$

$$p_1(d - d[bd(1-\alpha)]) = d_1d + d_2[bd(1-\alpha)]$$

$$p_1 = \frac{d_1d + d_2[bd(1-\alpha)]}{d - d[bd(1-\alpha)]} = \frac{d_1 + d_2[b(1-\alpha)]}{1 - [bd(1-\alpha)]}$$

From the same reasons as in case b, the intersection point belongs to S_2.

The intersection point is
$(p_1, p_2) = \left(\frac{d_1+d_2[b(1-\alpha)]}{1-[bd(1-\alpha)]}, dp_1 + d_2\right) = \left(\frac{d_1+d_2[b(1-\alpha)]}{1-[bd(1-\alpha)]}, d\frac{d_1+d_2[b(1-\alpha)]}{1-[bd(1-\alpha)]} + d_2\right)$ ■

We found two intersection points:

$$\hat{p} = (p_1, p_2) = \left(b\frac{d_2+d(1-\alpha)d_1}{[1-bd(1-\alpha)]} + d_1, \frac{d_2+d(1-\alpha)d_1}{[1-bd(1-\alpha)]}\right) \quad (28)$$

$$\tilde{p} = (p_1, p_2) = \left(\frac{d_1+d_2[b(1-\alpha)]}{1-[bd(1-\alpha)]}, d\frac{d_1+d_2[b(1-\alpha)]}{1-[bd(1-\alpha)]} + d_2\right) \quad (29)$$

In figure 3 you can see the intersection of the line $f(p) = 0$ and S_2 and the two intersection points which were calculated.

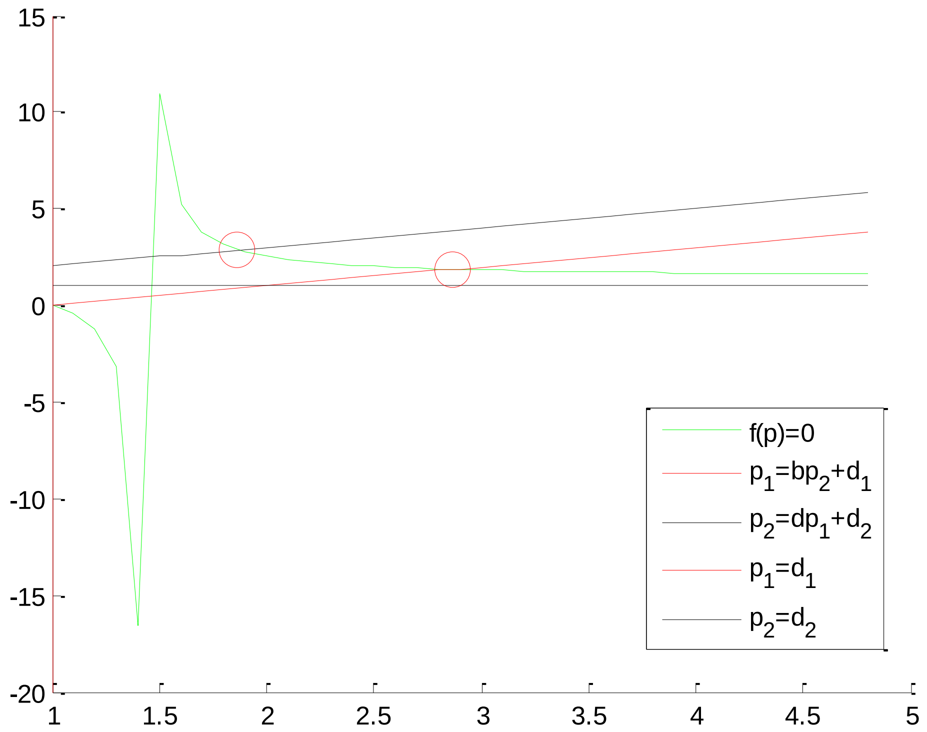

Figure 3,The red circles are the two intersection points

3.2.2 Case 2

The probability function in case 2 is $\frac{-d_1+p_1}{p_2 b}$ and we demand $\frac{-d_1+p_1}{p_2 b} \geq 1-\alpha$

$$\rightarrow -d_1 + p_1 \geq p_2 b(1-\alpha) \rightarrow p_2 \leq \frac{-d_1+p_1}{b(1-\alpha)} \quad (30)$$

And the conditions for case 2 are:

$$d_1 \leq p_1 \leq bp_2 + d_1 \; and \; p_2 > dp_1 + d_2$$

Definition:

Let $S_3 = \{(p_1, p_1) | d_1 \leq p_1 \leq bp_2 + d_1 \; and \; p_2 > dp_1 + d_2\}$

Let $f_3(p) = p_2 - \frac{-d_1+p_1}{b(1-\alpha)}$

In figure 4 you can see an example of the intersection of the line $f_3(p) = 0$ and S_3

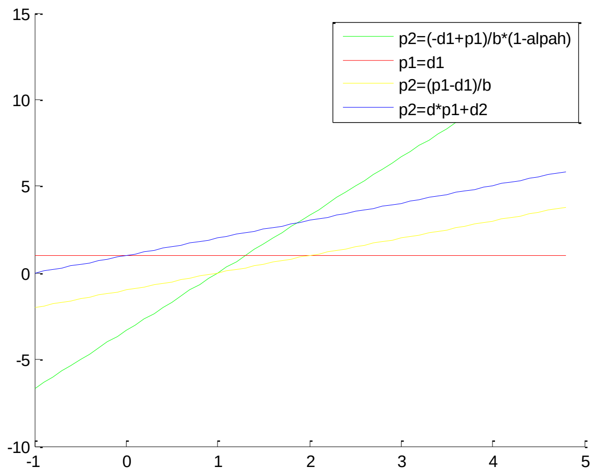

Figure 4

Finding the intersection points of the line $f_3(p) = 0$ and S_3:

Lemma 8: There is no intersection point between $f_3(p) = 0$ and $d_1 = p_1$ which belongs to S_3.

Proof :

$$\begin{cases} p_2 = \dfrac{-d_1 + p_1}{b(1-\alpha)} \\ d_1 = p_1 \end{cases}$$

$$p_2 = \frac{-d_1 + d_1}{b(1-\alpha)} = 0$$

In S_3 we demand $p_2 > dp_1 + d_2$ therefore $(p_1, p_2) = (d_1, 0) \notin S_3$. ■

Lemma 9: There is no intersection point between $f_3(p) = 0$ and $p_1 = bp_2 + d_1$ which belongs to S_3.

Proof :

$$\begin{cases} p_2 = \dfrac{-d_1 + p_1}{b(1-\alpha)} \\ p_1 = bp_2 + d_1 \end{cases}$$

$$p_2 = \frac{-d_1 + p_1}{b(1-\alpha)} = \frac{-d_1 + bp_2 + d_1}{b(1-\alpha)} = \frac{bp_2}{b(1-\alpha)} \rightarrow$$

$$\rightarrow p_2 b(1-\alpha) = bp_2 \rightarrow p_2(1-\alpha) - p_2 = 0 \rightarrow$$

$$p_2(1 - \alpha - 1) = 0 \rightarrow \alpha p_2 = 0 \rightarrow p_2 = 0$$

In S_3 we demand $p_2 > dp_1 + d_2$ therefore $(p_1, 0) = (d_1, 0) \notin S_3$. ■

Lemma 10: There is an intersection point between $f_3(p) = 0$ and $p_2 = dp_1 + d_2$ which belongs to S_3 and the point is $(p_1, p_2) = \left(\frac{d_2 b(1-\alpha)}{(1-db(1-\alpha))}, d\frac{d_2 b(1-\alpha)}{(1-db(1-\alpha))} + d_2\right)$.

$$\begin{cases} p_2 = \dfrac{-d_1 + p_1}{b(1-\alpha)} \\ p_2 = dp_1 + d_2 \end{cases}$$

$$p_2 = \frac{-d_1 + p_1}{b(1-\alpha)} = dp_1 + d_2 \rightarrow -d_1 + p_1 = dp_1 b(1-\alpha) + d_2 b(1-\alpha) \rightarrow$$

$$p_1 - dp_1 b(1-\alpha) = d_2 b(1-\alpha) + d_1 \rightarrow p_1\big(1 - db(1-\alpha)\big) = d_2 b(1-\alpha) + d_1 \rightarrow$$

$$p_1 = \frac{d_2 b(1-\alpha) + d_1}{\big(1 - db(1-\alpha)\big)}, p_2 = dp_1 + d_2 = d\frac{d_2 b(1-\alpha) + d_1}{\big(1 - db(1-\alpha)\big)} + d_2$$

The intersection point for case 2 is: $(p_1, p_2) = \left(\frac{d_2 b(1-\alpha)}{(1-db(1-\alpha))}, d\frac{d_2 b(1-\alpha)}{(1-db(1-\alpha))} + d_2\right)$ (31) ■

In figure 5 you can see the intersection of the line $f_3(p) = 0$ and S_3 and the intersection point which was calculated.

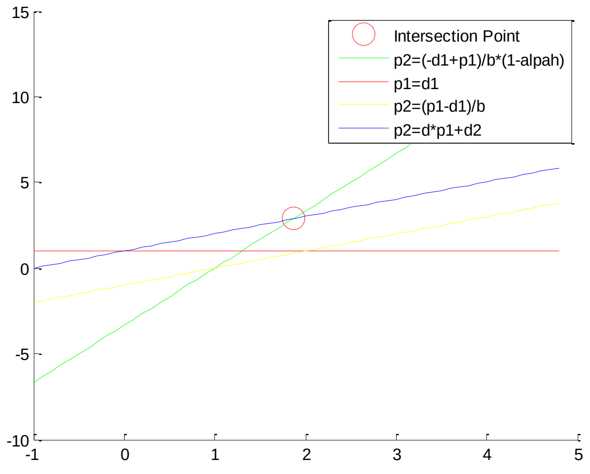

Figure 5

3.2.3 Case 3

The probability function in case 3 is $\frac{-d_2+p_2}{p_1 d}$ and we demand $\frac{-d_2+p_2}{p_1 d} \geq 1-\alpha$

$$\rightarrow -d_2 + p_2 \geq p_1 d(1-\alpha) \rightarrow p_2 \geq p_1 d(1-\alpha) + d_2 \quad (32)$$

And the conditions for case 3 are:

$$p_1 > bp_2 + d_1 \text{ and } d_2 \leq p_2 \leq dp_1 + d_2$$

Definition:

Let $S_4 = \{(p_1, p_1) | p_1 > bp_2 + d_1 \text{ and } d_2 \leq p_2 \leq dp_1 + d_2\}$

Let $f_4(p) = p_2 - p_1 d(1-\alpha) + d_2$

In figure 6 you can see an example of the intersection of the line $f_4(p) = 0$ and S_4

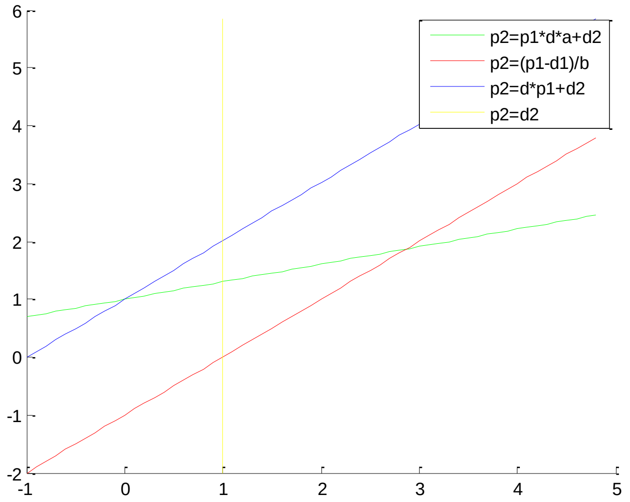

Figure 6

Finding the intersection points of the line $f_4(p) = 0$ and the condition set S_4:

Lemma 11: There is an intersection point between $f_4(p) = 0$ and $p_1 = bp_2 + d_1$ which belongs to S_3 and the point is $(p_1, p_2) = \left(b\frac{dd_1(1-\alpha)+d_2}{(1-db(1-\alpha))} + d_1, \frac{dd_1(1-\alpha)+d_2}{(1-db(1-\alpha))}\right)$.

Proof:

$$\begin{cases} p_2 = p_1 d(1-\alpha) + d_2 \\ p_1 = bp_2 + d_1 \end{cases}$$

$$p_2 = p_1 d(1-\alpha) + d_2 = (bp_2 + d_1)d(1-\alpha) + d_2 \rightarrow$$

$$p_2 = (bp_2 + d_1)d(1-\alpha) + d_2 \rightarrow$$

$$p_2 = dbp_2(1-\alpha) + dd_1(1-\alpha) + d_2 \rightarrow$$

$$p_2 - dbp_2(1-\alpha) = dd_1(1-\alpha) + d_2 \rightarrow$$

$$p_2\big(1 - db(1-\alpha)\big) = dd_1(1-\alpha) + d_2 \rightarrow$$

$$p_2 = \frac{dd_1(1-\alpha) + d_2}{\big(1 - db(1-\alpha)\big)}, p_1 = bp_2 + d_1 = b\frac{dd_1(1-\alpha) + d_2}{\big(1 - db(1-\alpha)\big)} + d_1 \blacksquare$$

<u>Lemma 12:</u> There is no intersection point between $f_4(p) = 0$ and $p_2 = d_2$ which belongs to S_4.

<u>Proof :</u>

$$\begin{cases} p_2 = p_1 d(1-\alpha) + d_2 \\ p_2 = d_2 \end{cases}$$

$$p_2 = p_1 d(1-\alpha) + d_2 = d_2 \rightarrow p_1 d(1-\alpha) = 0 \rightarrow$$

$$p_1 = 0$$

In S_4 we demand $p_1 > bp_2 + d_1$ therefore $(0, p_2) = (0, d_2) \notin S_4$. $\blacksquare$

<u>Lemma 13:</u> There is no intersection point between $f_4(p) = 0$ and $p_2 = dp_1 + d_2$ which belongs to S_4.

<u>Proof :</u>

$$\begin{cases} p_2 = p_1 d(1-\alpha) + d_2 \\ p_2 = dp_1 + d_2 \end{cases}$$

$$p_2 = p_1 d(1-\alpha) + d_2 = dp_1 + d_2 \rightarrow p_1 d(1-\alpha) = dp_1 \rightarrow$$

$$p_1 d(1-\alpha) - dp_1 = 0 \rightarrow p_1(d(1-\alpha) - d) = 0 \rightarrow$$

$$p_1 = 0$$

In S_4 we demand $p_1 > bp_2 + d_1$ therefore $(0, p_2) = (0, d_2) \notin S_4$. $\blacksquare$

The intersection point for case 3 is: $(p_1, p_2) = \left(b\frac{dd_1(1-\alpha)+d_2}{(1-db(1-\alpha))} + d_1, \frac{dd_1(1-\alpha)+d_2}{(1-db(1-\alpha))}\right)$(33)

Example of the line $p_2 = p_1 d(1-\alpha) + d_2$ with the conditions and the intersection point:

In figure 7 you can see the intersection of the line $f_4(p) = 0$ and S_4 and the intersection point which was calculated.

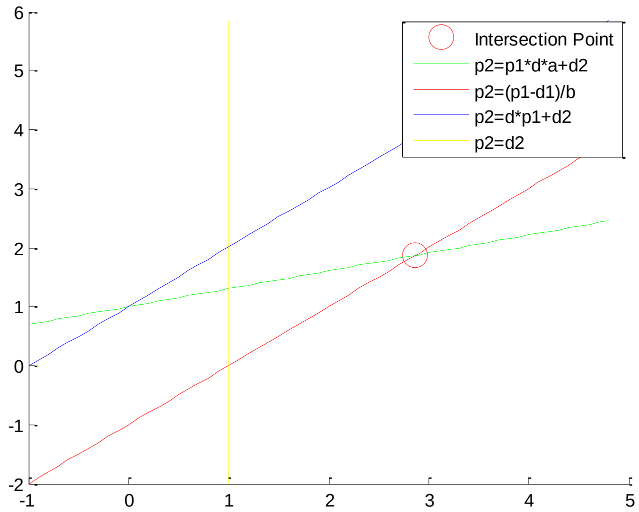

Figure 7

3.3 Solution and examples

Now that we have the intersection points for every case, we can find the solution for the minimum problem.

The problem:

$$\min c^T p$$

Subject to:

$$P(p - Cp \geq d) \geq 1 - \alpha$$

Where:

$c = \binom{c_1}{c_2}$, c_i is the cost for producing one unit of p_i.

And the probability function is:

$$\begin{cases} \frac{-d_1 + p_1}{p_2 b} \cdot \frac{-d_2 + p_2}{p_1 d} & , d_1 \leq p_1 \leq bp_2 + d_1 \ and \ d_2 \leq p_2 \leq dp_1 + d_2 \\ \frac{-d_1 + p_1}{p_2 b} & , d_1 \leq p_1 \leq bp_2 + d_1 \ and \ p_2 > dp_1 + d_2 \\ \frac{-d_2 + p_2}{p_1 d} & , p_1 > bp_2 + d_1 \ and \ d_2 \leq p_2 \leq dp_1 + d_2 \end{cases} \quad (19)$$

The solution will be in one of the cases:1,2,3. The case where the solution exists deepens on the cost coefficients.

Let us try to find the conditions for a solution in every one of the cases and the solution itself.

3.3.1 Case 1:

In this we case there are two intersection points:

$$\hat{p} = (p_1, p_2) = \left(b\frac{d_2+d(1-\alpha)d_1}{[1-bd(1-\alpha)]} + d_1, \frac{d_2+d(1-\alpha)d_1}{[1-bd(1-\alpha)]}\right) \quad (24)$$

$$\tilde{p} = (p_1, p_2) = \left(\frac{d_1+d_2[b(1-\alpha)]}{1-[bd(1-\alpha)]}, d\frac{d_1+d_2[b(1-\alpha)]}{1-[bd(1-\alpha)]} + d_2\right) \quad (25)$$

We will find the solution for the minimum problem where

$$f(p) = d_1d_2 - d_1p_2 - d_2p_1 + p_1p_2[1 - bd(1-\alpha)] = 0$$

Proof:

Let's assume that the minimum solution is the point (p_1, p_2) and that $f(p_1, p_2) > 0$

$$c_1p_1 + c_2p_2 = M$$

$$f(p) = d_1d_2 - d_1p_2 - d_2p_1 + p_1p_2[1 - bd(1-\alpha)] > 0 \rightarrow$$

$$p_1 > \frac{d_1p_2 - d_1d_2}{-d_2 + p_2\big(1 - bd(1-\alpha)\big)}$$

But , we can choose

$$p^*{}_1 = \frac{d_1p_2 - d_1d_2}{-d_2 + p_2\big(1 - bd(1-\alpha)\big)}$$

So $p_1 > p^*{}_1$

$$c_1p_1 + c_2p_2 - p^*{}_1c_1 - c_2p_2 = c_1\big(p_1 - p^*{}_1\big) > 0 \rightarrow c_1p^*{}_1 + c_2p_2 < M$$

And this is contradiction to the fact that M is the minimum of the function. ■

The solution will exists in this case if the cost coefficients and the probability function will have the same gradient.

A solution (p_1^*, p_2^*) will have to satisfy the condition:

$$\nabla f(p_1^*, p_2^*) || c$$

Lemma 14: The slope of the line $f(p) = 0$ is decreasing except the jump where the function in not defined.

<u>Proof:</u>

$$p_2 = \frac{d_2p_1 - d_1d_2}{-d_1 + p_1[1 - bd(1-\alpha)]}$$

$$p'_2 = \frac{d_2(-d_1 + p_1[1 - bd(1-\alpha)]) - (d_2p_1 - d_1d_2)[1 - bd(1-\alpha)]}{(-d_1 + p_1[1 - bd(1-\alpha)])^2} =$$

$$= \frac{-d_1d_2 + d_2p_1[1 - bd(1-\alpha)] - d_2p_1[1 - bd(1-\alpha)] + d_1d_2[1 - bd(1-\alpha)]}{(-d_1 + p_1[1 - bd(1-\alpha)])^2}$$

$$= \frac{d_1d_2[1 - bd(1-\alpha) - 1]}{(-d_1 + p_1[1 - bd(1-\alpha)])^2} = -\frac{d_1d_2bd(1-\alpha)}{(-d_1 + p_1[1 - bd(1-\alpha)])^2} \leq 0$$

The line $f(p) = 0$ is not defined in the point:

$$p_1{}^* = \frac{d_1}{1 - bd(1-\alpha)}$$

It is easy to see that in the two intersection points $\hat{p}, \tilde{p}$ $p_1 > p_1{}^*$ ■

<u>Conclusion from Lemma 14:</u> between the two intersection points the slope of the line $f(p) = 0$ is decreasing and doesn't change its direction.

A solution will exists in this case if the following inequality will be satisfied:

$$min\left(-\frac{f'_{p_1}(\tilde{p})}{f'_{p_2}(\tilde{p})}, -\frac{f'_{p_1}(\hat{p})}{f'_{p_2}(\hat{p})}\right) \leq -\frac{c_1}{c_2} \leq max\left(-\frac{f'_{p_1}(\tilde{p})}{f'_{p_2}(\tilde{p})}, -\frac{f'_{p_1}(\hat{p})}{f'_{p_2}(\hat{p})}\right) \quad (28)$$

Inequality (28) means that only if the Gradient of the cost coefficients is between Gradients of the probability function at the intersection points the solution can be found in the case otherwise the solution will be found in one of the other cases.

$$f(p) = d_1d_2 - d_1p_2 - d_2p_1 + p_1p_2[1 - bd(1-\alpha)]$$

$$f'_{p_1}(p) = -d_2 + p_2[1 - bd(1-\alpha)]$$

$$f'_{p_2}(p) = -d_1 + p_1[1 - bd(1-\alpha)]$$

$$f'_{p_1}(\tilde{p}) = -d_2 + \left(d\frac{d_1 + d_2[b(1-\alpha)]}{1 - [bd(1-\alpha)]} + d_2\right)[1 - bd(1-\alpha)]$$

$$= -d_2 + d(d_1 + d_2[b(1-\alpha)]) + d_2[1 - bd(1-\alpha)] =$$

$$= -d_2 + dd_1 + dd_2[b(1-\alpha)] + d_2 - d_2bd(1-\alpha) =$$

$$= dd_1 + dd_2[b(1-\alpha)] - d_2bd(1-\alpha) =$$

$$= dd_1 + bdd_2(1-\alpha) - d_2 bd(1-\alpha) = dd_1$$

$$f'_{p_2}(\tilde{p}) = -d_1 + \left(\frac{d_1 + d_2[b(1-\alpha)]}{1-[bd(1-\alpha)]}\right)[1 - bd(1-\alpha)] = -d_1 + d_1 + d_2[b(1-\alpha)]$$

$$= d_2[b(1-\alpha)] = d_2\, b(1-\alpha)$$

$$f'_{p_1}(\hat{p}) = -d_2 + \left(\frac{d_2 + d(1-\alpha)d_1}{[1-bd(1-\alpha)]}\right)[1 - bd(1-\alpha)] = -d_2 + d_2 + d(1-\alpha)d_1$$

$$= dd_1(1-\alpha)$$

$$f'_{p_2}(\hat{p}) = -d_1 + \left(b\frac{d_2 + d(1-\alpha)d_1}{[1-bd(1-\alpha)]} + d_1\right)[1 - bd(1-\alpha)] =$$

$$= -d_1 + b(d_2 + d(1-\alpha)d_1) + d_1[1 - bd(1-\alpha)] =$$

$$= -d_1 + bd_2 + db(1-\alpha)d_1 + d_1 - d_1 bd(1-\alpha)$$

$$= bd_2$$

Now we need to find out what is $max\left(-\frac{f'_{p_1}(\tilde{p})}{f'_{p_2}(\tilde{p})}, -\frac{f'_{p_1}(\hat{p})}{f'_{p_2}(\hat{p})}\right)$

$$-\frac{f'_{p_1}(\hat{p})}{f'_{p_2}(\hat{p})} + \frac{f'_{p_1}(\tilde{p})}{f'_{p_2}(\tilde{p})} = -\frac{dd_1(1-\alpha)}{bd_2} + \frac{dd_1}{d_2\, b(1-\alpha)} = \frac{dd_1 - dd_1(1-\alpha)^2}{d_2\, b(1-\alpha)}$$
$$= \frac{dd_1\left(1-(1-\alpha)^2\right)}{d_2\, b(1-\alpha)} > 0 \rightarrow -\frac{f'_{p_1}(\hat{p})}{f'_{p_2}(\hat{p})} > -\frac{f'_{p_1}(\tilde{p})}{f'_{p_2}(\tilde{p})}$$

Now after rewriting the inequality (28) we will get:

$$-\frac{dd_1}{d_2\, b(1-\alpha)} \leq -\frac{c_1}{c_2} \leq -\frac{dd_1(1-\alpha)}{bd_2} \qquad (29)$$

If the inequality (29) is satisfied then the solution can be found in case 2 , otherwise the solution will be found in one of the other cases.

To get the solution we have to solve these equations:

$$\begin{cases} -\frac{c_1}{c_2} = -\frac{f'_{p_1}(p)}{f'_{p_2}(p)} & (30) \\ f(p) = d_1d_2 - d_1p_2 - d_2p_1 + p_1p_2[1 - bd(1-\alpha)] = 0 & (31) \end{cases}$$

$$f'_{p_1}(p) = -d_2 + p_2[1 - bd(1-\alpha)]$$

$$f'_{p_2}(p) = -d_1 + p_1[1 - bd(1-\alpha)]$$

The solutions are:

$$(p_{1_1}, p_{1_2}) = \left(-\frac{c_1 d_1 + c_2 \sqrt{\frac{(1-\alpha) b d d_1 d_2 c_1}{c_2}}}{c_1 [bd(1-\alpha) - 1]}, -\frac{d_2 + \sqrt{\frac{(1-\alpha) b d d_1 d_2 c_1}{c_2}}}{bd(1-\alpha) - 1} \right)$$

$$(p_{2_1}, p_{2_2}) = \left(-\frac{c_1 d_1 - c_2 \sqrt{\frac{(1-\alpha) b d d_1 d_2 c_1}{c_2}}}{c_1 [bd(1-\alpha) - 1]}, -\frac{d_2 - \sqrt{\frac{(1-\alpha) b d d_1 d_2 c_1}{c_2}}}{bd(1-\alpha) - 1} \right)$$

Example 1:

Let's take a look on the example parameters:

$$d_1 = d_2 = 1, \alpha = 0.7, b = 1, d = 1.01, c_1 = 0.8, c_2 = 0.9$$

The conditions we demand for the solution to be in this case are:

$$\begin{cases} bd \geq 0 \\ 1 - bd(1-\alpha) > 0 \\ -\frac{dd_1}{d_2\, b(1-\alpha)} \leq -\frac{c_1}{c_2} \leq -\frac{dd_1(1-\alpha)}{bd_2} \end{cases}$$

We will verify it:

$$bd \geq 0 \to 1 * 1.01 = 1.01 > 1$$

$$1 - bd(1-\alpha) > 0 \to 1 - 1 * 1.01 * 0.3 = 0.697 > 0$$

$$-\frac{dd_1}{d_2\, b(1-\alpha)} \leq -\frac{c_1}{c_2} \leq -\frac{dd_1(1-\alpha)}{bd_2} \to -\frac{1.01}{0.3} \leq -\frac{0.8}{0.9} \leq -\frac{0.303}{1} \to$$

$$-3.3667 \leq -0.8889 \leq -0.303$$

All the conditions are satisfied.

In figure 8 you can see the line $f(p) = 0$, the red and blue circles are points which belongs to S_2

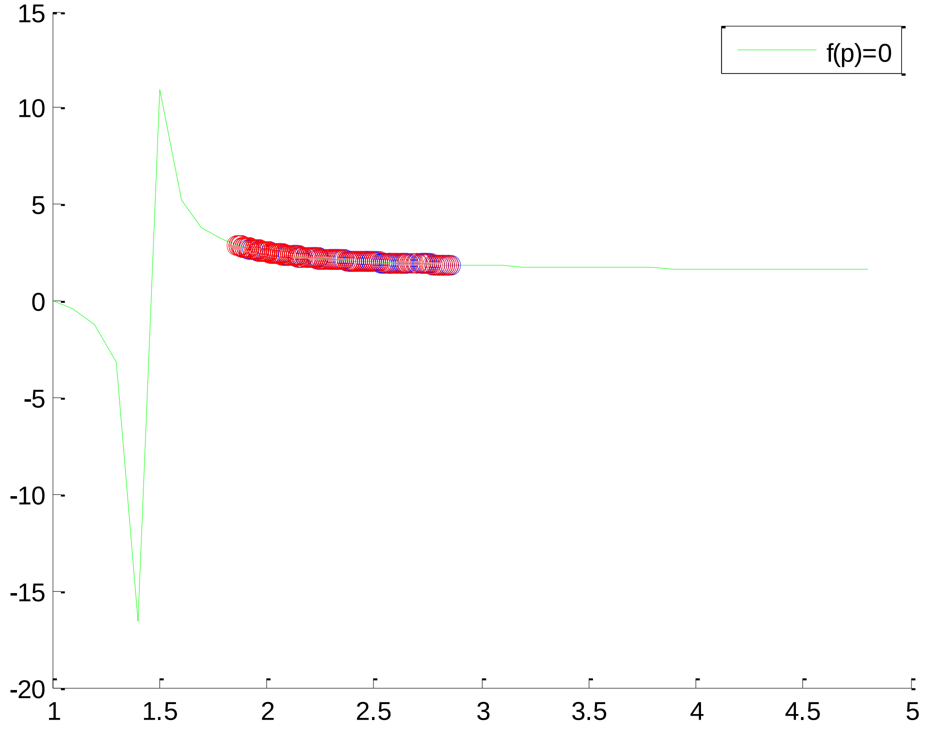

Figure 8, The red and blue circles are points which belongs to S2 and on the line f(p)=0

The green line is the probability function and the red and blue points are possible soultion in which the probability is higer then 0.3.

Now we will add to the figure the conditions for case 2:

$$\begin{cases} p_1 = bp_2 + d_1 \\ p_2 = dp + d_2 \end{cases}$$

In figure 9 you can see the line $f(p) = 0$, the red and blue circles are points which belongs to S_2 and the lines which creates S_2

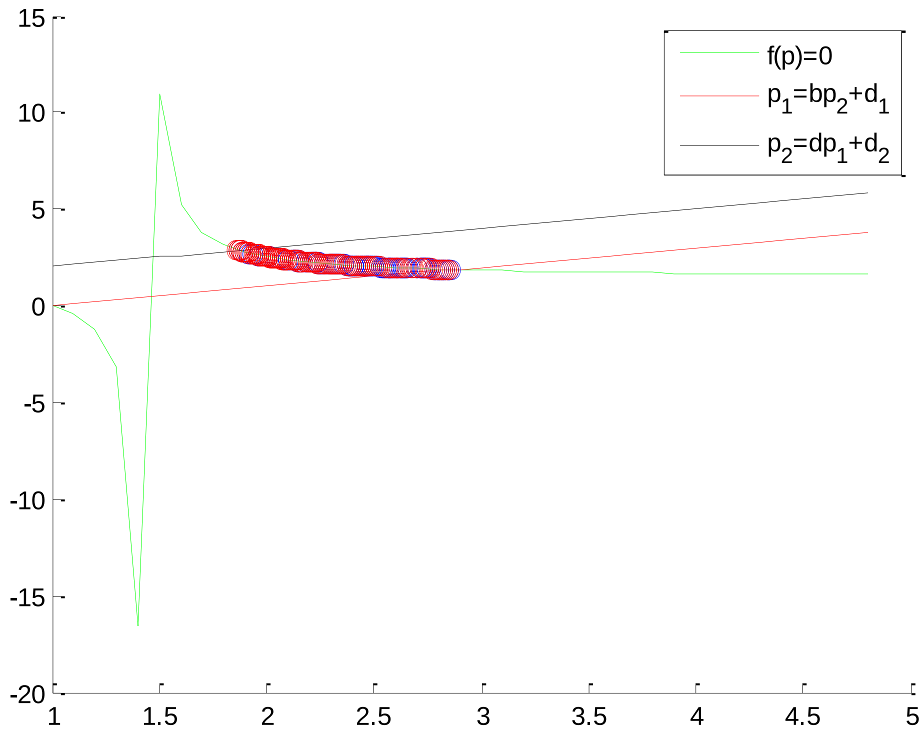

Figure 9,The red and blue circles are points which belongs to S2 and on the line f(p)=0

The black line is the condition $p_2 = dp + d_2$ and the red line is the condition $p_1 = bp_2 + d_1$.

In figure 10 we are zooming in to the intercating zone between the line $f(p) = 0$ and the condition set S_2:

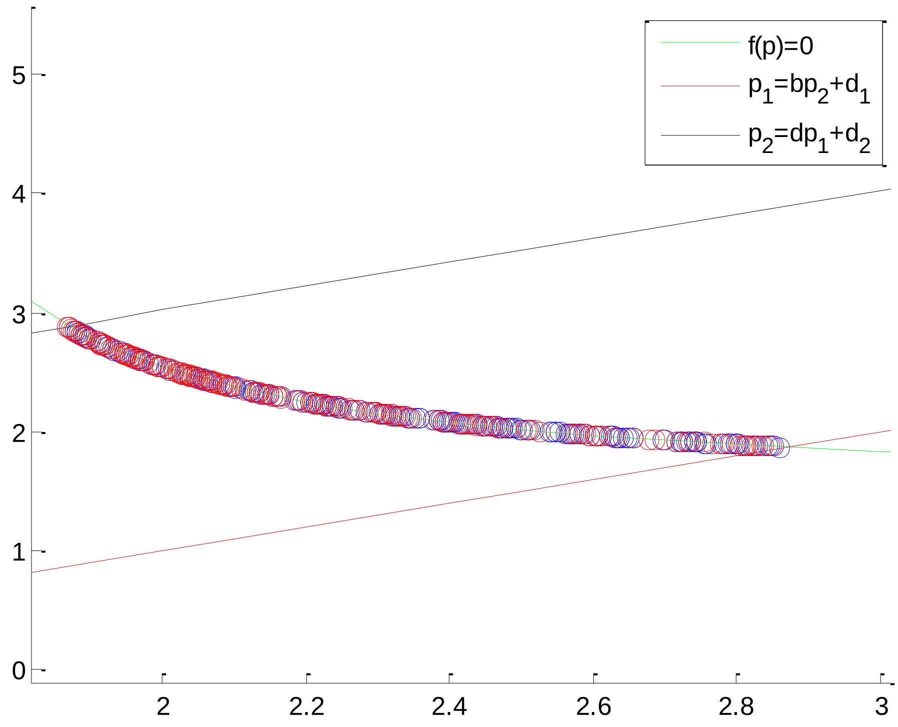

Figure 10,The red and blue circles are points which belongs to S2 and on the line f(p)=0

The soultion we can calcuate is :

$$(p_{1_1}, p_{1_2}) = \left(-\frac{c_1 d_1 + c_2 \sqrt{\frac{(1-\alpha) b d d_1 d_2 c_1}{c_2}}}{c_1 [bd(1-\alpha) - 1]}, -\frac{d_2 + \sqrt{\frac{(1-\alpha) b d d_1 d_2 c_1}{c_2}}}{bd(1-\alpha) - 1} \right) =$$

$$= (2.2723, 2.1793)$$

We will add the soultion to figure 10. The soultion is the point in blue in figure 11:

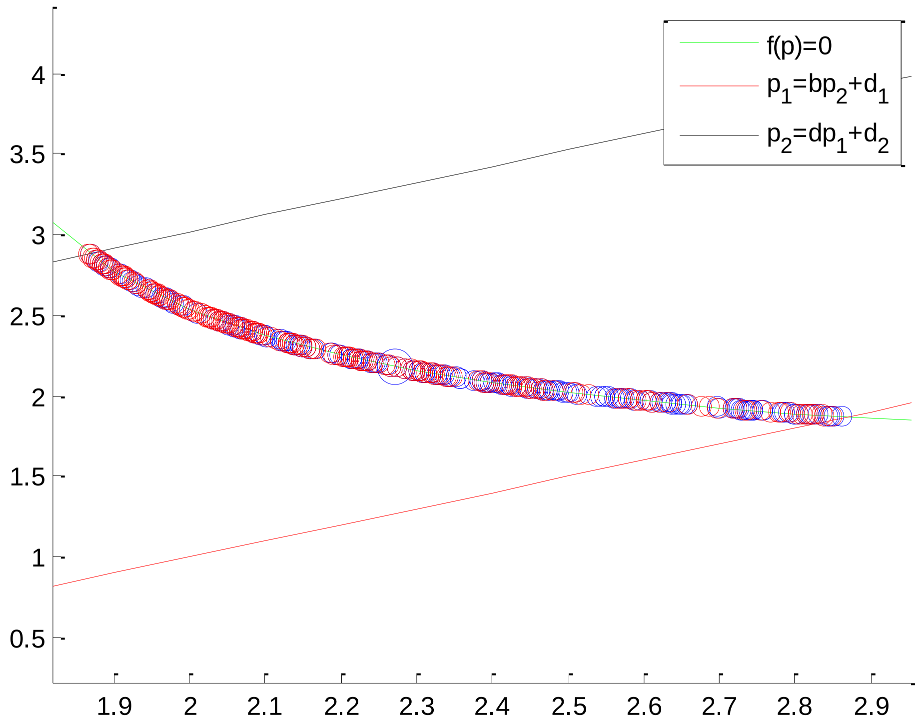

Figure 11, The red and blue circles are points which belongs to S2 and on the line f(p)=0. The larger blue circle is the solution.

And finally will add the cost function and will see in figure 12 the soultion in blue , possible soultion in red and blue , intersection lines and points and the cost functioin :

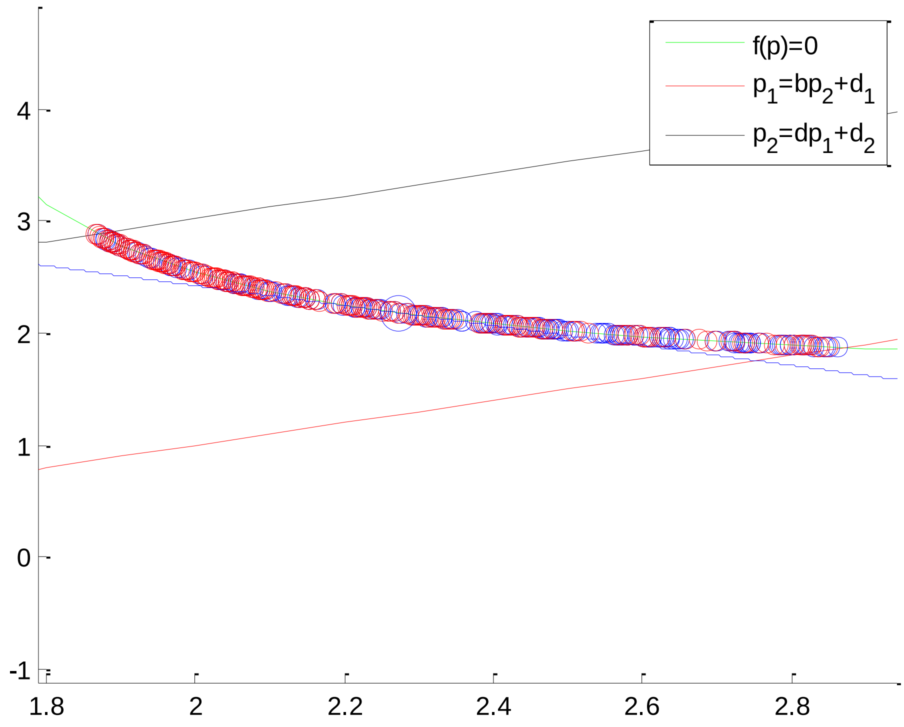

Figure 12 , The red and blue circles are points which belongs to S2 and on the line f(p)=0. The larger blue circle is the solution. The blue line is the cost function.

3.3.2 Case 2:

In this we case there is one intersection point:

$$(p_1, p_2) = \left(\frac{d_2 b(1-\alpha)}{(1-db(1-\alpha))}, d\frac{d_2 b(1-\alpha)}{(1-db(1-\alpha))} + d_2\right) \quad (32)$$

In this case the probability function is linear therefor the solution will be in the intersection point between the probability function and the condition; the solution will be point (32).

A solution will exists in this case if the following inequality will be satisfied:

$$-\frac{dd_1}{d_2\, b(1-\alpha)} > -\frac{c_1}{c_2}$$

Example 2:

Let's take a look on the example parameters:

$$d_1 = d_2 = 1, \alpha = 0.7, b = 1, d = 1.01, c_1 = 4, c_2 = 0.9$$

The conditions we demand for the solution to be in this case are:

$$\begin{cases} bd \geq 1 \\ 1 - bd(1-\alpha) > 0 \\ -\dfrac{dd_1}{d_2\, b(1-\alpha)} > -\dfrac{c_1}{c_2} \end{cases}$$

We will verify it:

$$bd \geq 1 \rightarrow 1 * 1.01 = 1.01 > 1$$

$$1 - bd(1-\alpha) > 0 \rightarrow 1 - 1 * 1.01 * 0.3 = 0.697 > 0$$

$$-\frac{dd_1}{d_2\, b(1-\alpha)} > -\frac{c_1}{c_2} \rightarrow -\frac{1.01}{0.3} > -\frac{4}{0.9}$$

$$-3.3667 > 4.4444$$

All the conditions are satisfied.

In figure 13 you can see the line $f_3(p) = 0$ and the condition set S_3 :

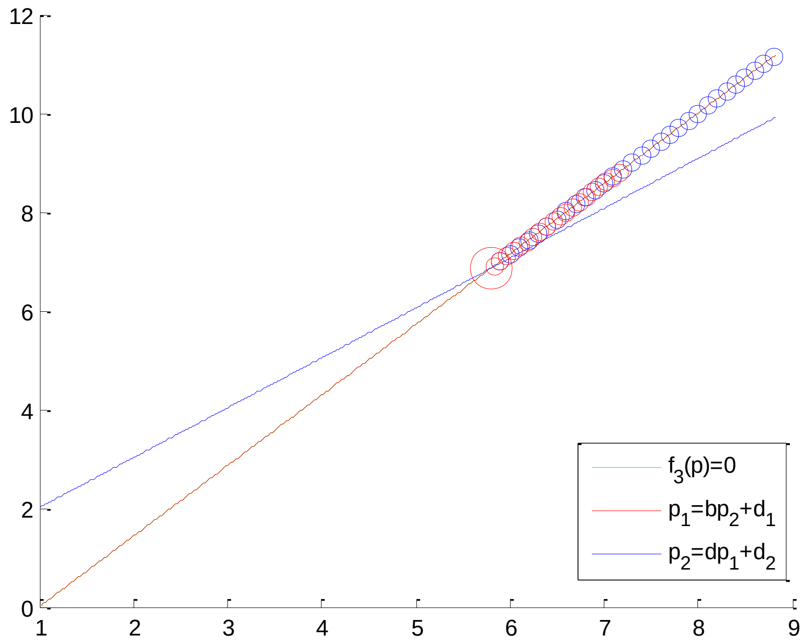

Figure 13 , The red and blue circles are points which belongs to S3 and on the line f(p)=0. The larger red circle is the solution.

The big red point is the solution. The small red and blur points are other possible solutions.

In figure 14 you can see the cost function with the line $f_3(p) = 0$ and the condition set S_3:

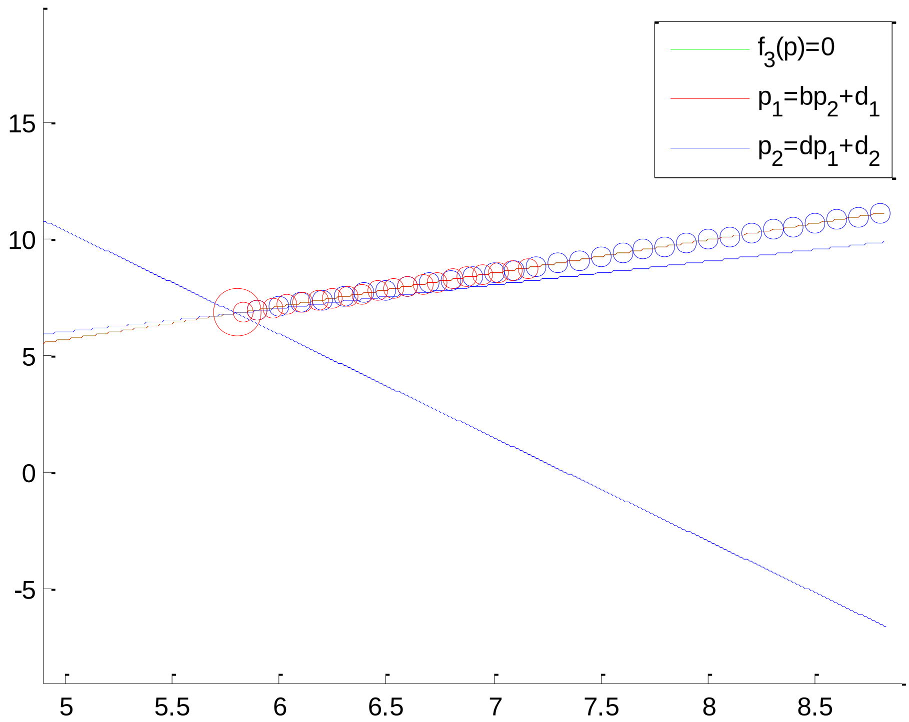

Figure 14 , The red and blue circles are points which belongs to S3 and on the line f(p)=0. The larger red circle is the solution. The blue line is the cost function.

3.3.3 Case 3:

In this we case there is one intersection point:

$$(p_1, p_2) = \left(b\frac{dd_1(1-\alpha)+d_2}{(1-db(1-\alpha))} + d_1, \frac{dd_1(1-\alpha)+d_2}{(1-db(1-\alpha))}\right) \quad (33)$$

In this case the probability function is linear therefor the solution will be in the intersection point between the probability function and the condition; the solution will be point (33).

A solution will exists in this case if the following inequality will be satisfied:

$$-\frac{c_1}{c_2} > -\frac{dd_1(1-\alpha)}{bd_2}$$

Example:

Let's take a look on the example parameters:

$$d_1 = d_2 = 1, \alpha = 0.7, b = 1, d = 1.01, c_1 = 0.8, c_2 = 4$$

The conditions we demand for the solution to be in this case are:

$$\begin{cases} bd \geq 1 \\ 1 - bd(1-\alpha) > 0 \\ -\dfrac{c_1}{c_2} > -\dfrac{dd_1(1-\alpha)}{bd_2} \end{cases}$$

We will verify it:

$$bd \geq 0 \to 1 * 1.01 = 1.01 > 1$$

$$1 - bd(1-\alpha) > 0 \to 1 - 1 * 1.01 * 0.3 = 0.697 > 0$$

$$-\frac{0.8}{4} > -\frac{0.303}{1} \to -0.2 > -0.303$$

All the conditions are satisfied.

In figure 15 you can see the line $f_4(p) = 0$ and the condition set S_4 :

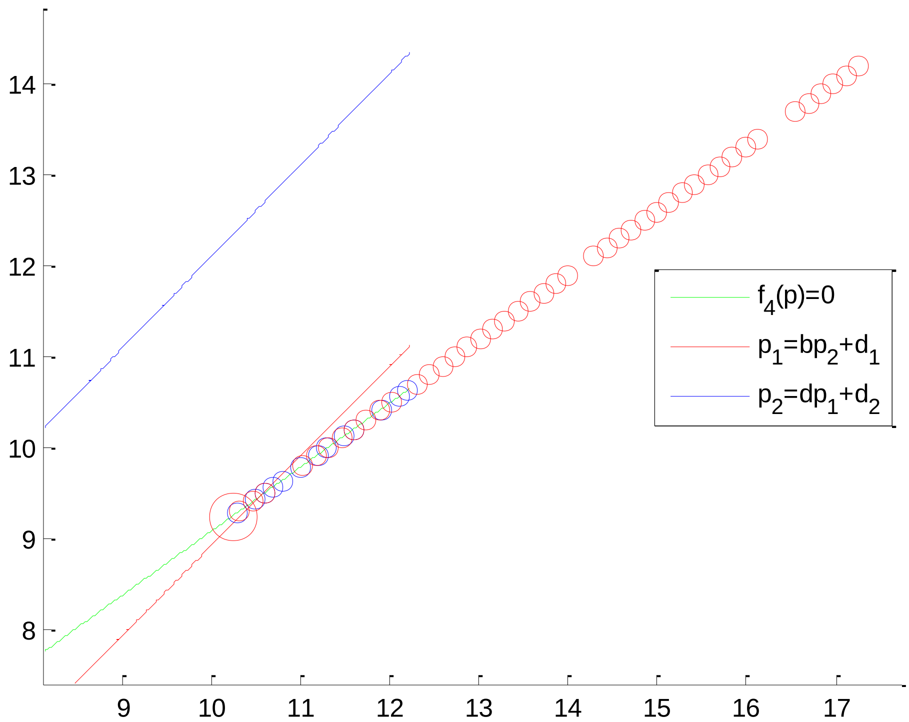

Figure 15, The red and blue circles are points which belongs to S4 and on the line f(p)=0. The larger red circle is the solution.

The big red point is the solution. The small red and blur points are other possible solutions.

In figure 16 you can see the cost function with the line $f_4(p) = 0$ and the condition set S_4:

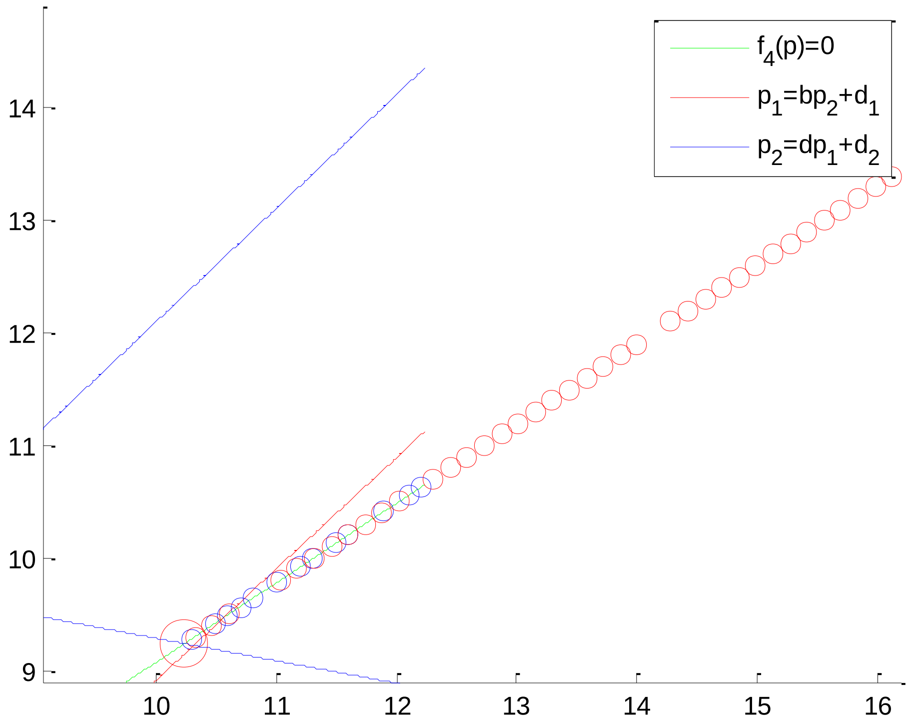

Figure 16 , The red and blue circles are points which belongs to S4 and on the line f(p)=0. The larger red circle is the solution. The blue line is the cost function.

4. Conclusions and future work

In this project we showed that even if the number of units produced by sector Si to produce one unit of sector Sj is a random variable the minimization problem for stochastic input-output model can be solved, under some assumptions.

If the random variables have known distribution function the probability function can be calculated and the minimization problem can be solved.

In this project we examined only specific case where the random variables are uniform distributed random variables, but more study can be done when other or mixed distribution functions are used.

In addition in this project we selected the number of products to be 2, more work can be done on a different number of products.

Appendix: MATLAB programs

1. **Experimental_testing_1_a.m used in page 9 :**

```
function [result] =experimental_testing_1_a(option)
b=1;
d1=1;
d2=1;
d=1.01;
Greater = 0;
Less=0;
if(option == 1)
   p1 = 0.5;
   p2= 0.9;
   calc_prob =0
   a1=b*p2+d1;
   a2=d*p1+d2;
   if((p1<d1) && (p2<d2))
      test = fprintf('Good\n');
   end
elseif (option == 2)
   p1 = 1.5;
   p2=1.9;
   calc_prob =((-d1+p1)/(p2*b))*((-d2+p2)/(p1*d))
   a1=b*p2+d1;
   a2=d*p1+d2;
   if((p1>=d1) && (p2>=d2) && (p1<=a1) && (p2<=a2))
      test = fprintf('Good\n');
   end
elseif (option == 3)
   p1 = 1.5;
   p2=5;
   calc_prob =((-d1+p1)/(p2*b))
   a1=b*p2+d1;
   a2=d*p1+d2;
   if((p1>=d1) && (p1<=a1) && (p2>a2))
      test = fprintf('Good\n');
   end
elseif(option ==4)
   p1 = 5;
   p2=1.5;
   calc_prob =((-d2+p2)/(p1*d))
   a1=b*p2+d1;
   a2=d*p1+d2;
   if( (p2>=d2) && (p1>a1) && (p2<=a2))
      test = fprintf('Good\n');
   end
end
%100000000
for I=1:100000000
   m12 = rand(1)*b;
   m21 = rand(1)*d;
   cond1 = p1-m12*p2;
```

```
    cond2 = p2-m21*p1;
    if ((cond1>=d1) && (cond2>=d2))
        Greater=Greater+1;
    else
        Less = Less+1;
    end
end
real_prob = Greater/(Greater+Less)
result = real_prob-calc_prob;
```

2. Experimental_testing_1_b.m used in page 13

```
function [result] =experimental_testing_1_b(option)
b=1;
d1=1;
d2=1;
d=1.01;
Greater = 0;
Less=0;
%p1 = 3.5;
p2=10;
result=0;
%1-bd(1-a)=0 -> 1-bd+abd=0-> abd=1-bd -> a= (bd-1)/bd
if (option == 1)
    a= (b*d-1)/(b*d);
elseif (option == 2)
    a=0.2;
    %a=0.5;
    %a=0.1;
else
    a=0.0096;
end
e=1-b*d*(1-a)
p1_req = (d1*d2*(1-e)*b*p2*d2*e)/(e*(b*p2*d+d1*d))
%100000000
for I=1:100000000
    p1_req = (d2*d1*(1-e))/(e*d*(p2+d1));
    %p1=rand(1)*(b*p2+d1-p1_req)+p1_req
    a1=b*p2+d1;
    p1=rand(1)*(a1-d1)+d1;
    a2=d*p1+d2;
    while (( p1 <d1) || (p2 < d2) || (p1 > a1) || (p2 >a2))
        %p1=rand(1)*(b*p2+d1-p1_req)+p1_req;
        a1=b*p2+d1;
        p1=rand(1)*(a1-d1)+d1;
        a2=d*p1+d2;
    end
    if((p1>=d1) && (p2>=d2) && (p1<=a1) && (p2<=a2))
        result=result+1;
    end
    calc_prob =((-d1+p1)/(p2*b))*((-d2+p2)/(p1*d));
    func=d1*d2-d1*p2-d2*p1+p1*p2*e;
    if ((calc_prob >= (1-a)) || (func >=0))
        Greater = Greater +1;
```

```
    else
        Less=Less+1;
    end
end
Greater
Less
sum=Greater+Less
```

3. **Programs for plotting case 1 :**

```
function plot_h(min_x1,min_x2,max_x1,max_x2,a,b,d,d1,d2,c1,c2)
x=min_x1:0.1:max_x1;
size1=size(x,2);
for I=1:size1
    y(I) = (d1*d2-d2*x(I))/(d1-x(I)*(1-a*b*d));
    y_3(I) = (x(I)-d1)/b;
    y_5(I) = d*x(I)+d2;
    con1(I) = d1;
    con2(I) = d2;
end
y1 = (d2+d*a*d1)/(1-b*d*a);
x1 = b*y1+d1;
x2 = (d1+d2*b*a)/(1-b*d*a);
y2 = d*x2+d2;
hold on;
plot(x,y,'-g');
plot(x,y_3,'-r');
plot(x,y_5,'-k');
yL = get(gca,'YLim');
%line([d1 d1],yL,'Color','r');
%plot(x,con2,'-k');
%plot(x1,y1,'or','MarkerSize',14)
%plot(x2,y2,'or','MarkerSize',14)
%legend('f(p)=0','p_1=bp_2+d_1','p_2=dp_1+d_2','p_1=d_1','p_2=d_2');
legend('f(p)=0','p_1=bp_2+d_1','p_2=dp_1+d_2');

function plot_option_2(min_x1,min_x2,max_x1,max_x2,a,b,d,d1,d2,c1,c2)
[X,Y] = meshgrid(min_x1:0.01:max_x1,min_x2:0.01:max_x2);
x = min_x1:0.001:max_x1;
y=min_x2:0.001:max_x2;
size1=size(X,1);
size2=size(X,2);
size3 = size(x,2);
size4 = size(y,2);
min_prob=-1;
min_x=-1;
min_y=-1;
min_func=-1;
max_prob = -1;
for I=1:size3
        x2 = (d1*d2-d2*x(I))/(d1-x(I)*(1-a*b*d));
        if x2>=d2 && x2<=(d*x(I)+d2) && d1<=x(I) && x(I)<=(b*x2+d1)
            [prob]=secoundprob(x(I),x2,d1,d2,b,d);
        else
```

```
            prob=0;
        end
        if (prob > max_prob)
            max_prob = prob;
        end
        if(prob >=a)
            plot(x(I),x2,'o','MarkerSize',8);
            hold on;
            func = c1*x(I)+c2*x2;
            if(min_func==-1)
                min_prob=prob;
                min_func=func;
                min_x=x(I);
                min_y=x2;
            elseif func<min_func
                min_func=func;
                min_prob=prob;
                min_x=x(I);
                min_y=x2;
            end
        end
end
for I=1:size4
        x1 = (d1*d2-d1*y(I))/(d2-y(I)*(1-a*b*d));
        if d1<=x1 && x1<=(b*y(I)+d1) && y(I)>=d2 && y(I)<=(d*x1+d2)
            [prob]=secoundprob(x1,y(I),d1,d2,b,d);
        else
            prob=0;
        end
        if (prob > max_prob)
            max_prob = prob;
        end
        if(prob >=a)
            plot(x1,y(I),'or','MarkerSize',8);
            hold on;
            func = c1*x1+c2*y(I);
            if(min_func==-1)
                min_prob=prob;
                min_func=func;
                min_x=x1;
                min_y=y(I);
            elseif func<min_func
                min_func=func;
                min_prob=prob;
                min_x=x1;
                min_y=y(I);
            end
        end
end
w=1-a*b*d;
item1 = -c1*w*(1-d1)+2*d2*w*c2;
item2=c1^2*w^2*(1-d1)^2+4*c2^2*w^2*d2*(d2-1)+d*c1*c2*w^2*(1-d1+d1*d2*(1-w));
```

```
item3 = 2*w^2*c2;
sol1_x2 = -(d2 + ((a*b*c1*d*d1*d2)/c2)^(1/2))/(a*b*d - 1);
sol2_x2 =  -(d2 - ((a*b*c1*d*d1*d2)/c2)^(1/2))/(a*b*d - 1);
sol1_x1= -(c1*d1 + c2*((a*b*c1*d*d1*d2)/c2)^(1/2))/(c1*(a*b*d - 1));
sol2_x1=-(c1*d1 - c2*((a*b*c1*d*d1*d2)/c2)^(1/2))/(c1*(a*b*d - 1));
[prob]=secoundprob(sol1_x1,sol1_x2,d1,d2,b,d);
check_x_option(sol1_x1,sol1_x2,d1,d2,b,d);
%[prob]=secoundprob(sol2_x1,sol2_x2,d1,d2,b,d)
%check_x_option(sol2_x1,sol2_x2,d1,d2,b,d)
func = c1*sol1_x1+c2*sol1_x2;
func = c1*min_x+c2*min_y;
hold on;
plot(sol1_x1,sol1_x2,'ob','MarkerSize',14)
hold on;
%plot(min_x,min_y,'or','MarkerSize',14)
plot_c1_c2(d1,d2,max_x1,max_x2,c1,c2,sol1_x1,sol1_x2);
fprintf('min_prob-%d,min_x-%d,min_y-%d,func-%d,max_prob-%d\nsol1_x1-%d,sol1_x2-
%d\nsol2_x1-%d,sol2_x2-
%d\n',min_prob,min_x,min_y,func,max_prob,sol1_x1,sol1_x2,sol2_x1,sol2_x2);
```

4. **<u>Programs for plotting case 2 :</u>**

```
function plot_option_3(min_x1,min_x2,max_x1,max_x2,a,b,d,d1,d2,c1,c2)
x = min_x1:0.01:max_x1;
size1 = size(x,2);
for I=1:size1
  y(I) = (-d1+x(I))/(a*b);
  y1(I) = (x(I)-d2)/(a*b);
  y2(I) = d*x(I)+d2;
end
hold on;
plot(x,y,'-g',x,y1,'-r',x,y2,'-b');
legend('f_3(p)=0','p_1=bp_2+d_1','p_2=dp_1+d_2');
function plot_option_3_con_test(min_x1,min_x2,max_x1,max_x2,a,b,d,d1,d2,c1,c2)
x = min_x1:0.1:max_x1;
y=min_x2:0.1:max_x2;
size3 = size(x,2);
size4 = size(y,2);
min_prob=-1;
min_x=-1;
min_y=-1;
min_func=-1;
max_prob = -1;
hold on;
for I=1:size3
        x2 = (-d1+x(I))/(a*b);
        if a*x2+d1-x(I)<=0 && x(I)-((x2-d2)/d)<0
            [prob]=secoundprob(x(I),x2,d1,d2,b,d);
        else
            prob=0;
        end
        if (prob > max_prob)
            max_prob = prob;
        end
```

```
        if(prob >=a)
            plot(x(I),x2,'ob','MarkerSize',8);
            hold on;
            func = c1*x(I)+c2*x2;
            if(min_func==-1)
                min_prob=prob;
                min_func=func;
                min_x=x(I);
                min_y=x2;
            elseif func<min_func
                min_func=func;
                min_prob=prob;
                min_x=x(I);
                min_y=x2;
            end
        end
end
for I=1:size4
        x1 = a*y(I)*b+d1;
        if a*y(I)+d1-x1<=0 && x1-((y(I)-d2)/d)<0
            [prob]=secoundprob(x1,y(I),d1,d2,b,d);
        else
            prob=0;
        end
        if (prob > max_prob)
            max_prob = prob;
        end
        if(prob >=a)
            plot(x1,y(I),'ob','MarkerSize',8);
            hold on;
            func = c1*x1+c2*y(I);
            if(min_func==-1)
                min_prob=prob;
                min_func=func;
                min_x=x1;
                min_y=y(I);
            elseif func<min_func
                min_func=func;
                min_prob=prob;
                min_x=x1;
                min_y=y(I);
            end
        end
end
func = c1*min_x+c2*min_y;
fprintf('min_prob-%d,min_x-%d,min_y-%d,func-%d,max_prob-
%d\n',min_prob,min_x,min_y,func,max_prob);
```

5. Programs for plotting case 3:

```
function plot_option_4(min_x1,min_x2,max_x1,max_x2,a,b,d,d1,d2,c1,c2)
x = min_x1:0.01:max_x1;
size1 = size(x,2);
for I=1:size1
   y(I) = a*x(I)*d+d2;
   y1(I)=(x(I)-d1)/d;
   y2(I)=d*x(I)+d2;
end
hold on;
plot(x,y,'-g',x,y1,'-r',x,y2,'-b');
legend('f_4(p)=0','p_1=bp_2+d_1','p_2=dp_1+d_2');

function plot_option_4_con_new(min_x1,min_x2,max_x1,max_x2,a,b,d,d1,d2,c1,c2)
x = min_x1:0.1:max_x1;
y=min_x2:0.1:max_x2;
size3 = size(x,2);
size4 = size(y,2);
min_prob=-1;
min_x=-1;
min_y=-1;
min_func=-1;
max_prob = -1;
hold on;
for I=1:size3
      x2 = a*x(I)*d+d2;
      if x2>=d2 && x2<=(d*x(I)+d2) && x(I)>(b*x2+d1)
         [prob]=secoundprob(x(I),x2,d1,d2,b,d);
      else
         prob=0;
      end
      if (prob > max_prob)
         max_prob = prob;
      end
      if(prob >=a)
         plot(x(I),x2,'o','MarkerSize',8);
         hold on;
         func = c1*x(I)+c2*x2;
         if(min_func==-1)
            min_prob=prob;
            min_func=func;
            min_x=x(I);
            min_y=x2;
         elseif func<min_func
            min_func=func;
            min_prob=prob;
            min_x=x(I);
            min_y=x2;
         end
      end
end
for I=1:size4
```

```
        x1 = (y(I)-d2)/(a*d);
        if y(I)>=d2 && y(I)<=(d*x1+d2) && x1>(b*y(I)+d1)
            [prob]=secoundprob(x1,y(I),d1,d2,b,d);
        else
            prob=0;
        end
        if (prob > max_prob)
            max_prob = prob;
        end
        if(prob >=a)
            plot(x1,y(I),'or','MarkerSize',8);
            hold on;
            func = c1*x1+c2*y(I);
            if(min_func==-1)
                min_prob=prob;
                min_func=func;
                min_x=x1;
                min_y=y(I);
            elseif func<min_func
                min_func=func;
                min_prob=prob;
                min_x=x1;
                min_y=y(I);
            end
        end
end
func = c1*min_x+c2*min_y;
fprintf('min_prob-%d,min_x-%d,min_y-%d,func-%d,max_prob-
%d\n',min_prob,min_x,min_y,func,max_prob);
```