אורט בראודה
המכללה האקדמית להנדסה
ORT Braude College

המחלקה למתמטיקה
Department of Mathematics

פרויקט מסכם לתואר בוגר במדעים (B.Sc) במתמטיקה שימושית

הימורים אופטימליים ע"י שימוש בקריטריון קלי

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2014

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1 Introduction

When gambling, the gambler has to decide how much of his capital to gamble in a bet. For example, if the gambler chooses too much capital to wager in a high risk bet, he could potentially suffer a catastrophic loss. If he bets too little in a low risk bet, he could be missing out on growing his capital even more. Consider a simple scenario of a gambler playing a favorable betting game which contains a long series of bets (by favorable meaning the odds of each single bet are in the gambler's favor). He could, for example, just choose to bet everything he has on each single bet in the series because the game is favorable to him. But after playing a long series of bets, the gambler will almost surely suffer a catastrophic failure. Therefore the gambler needs a way to manage his capital and determine the right amount of capital to allocate to a bet which consequently also helps him lower his risk of catastrophic failure for a series of bets.

The purpose of this paper is to present the Kelly Criterion which provides a formula for determining how to choose a fraction of capital to wager in a bet. The Kelly Criterion is applicable to any kind of game where a gambler has to make a bet, for example a game can be a gamble in the stock market, a horse race or a card game.

The Kelly Criterion was formulated by the scientist John Larry Kelly Jr. who worked for Bell Labs in 1956. Kelly developed a method for betting in games of roulette and blackjack and for investing in the stock market that was based on the principles of information theory which was eventually called the Kelly Criterion. Today the Criterion is part of mainstream investment theory.

The way the Kelly Criterion works is it maximizes what is called a utility function. In our paper we will define utility as the long term capital growth rate the gambler gets if he bets a specific fraction of capital in the bet. For example, we will prove later on that the utility function of a gambler playing a game of coin toss is a logarithmic function $g(f)$ where f is the fraction of capital wagered in a coin toss bet.

We will show that by maximizing the utility function the gambler can find the optimal amount of capital to wager in the bet. This will allow him to better manage his money. We will also show that as a consequence of using the Kelly Criterion the gambler will lower the risk of catastrophic failure to zero.

We will first illustrate a simple and naive way of determining the optimal fraction of betting in a game of coin toss. We will then determine the optimal fraction of betting using the Kelly Criterion for a game of coin toss and analyze different games that are variations of the coin toss game. Finally we will describe how to apply the Kelly Criterion to more complex games, such as Sports Betting and the Stock Market.

2 Binomial Games

2.1 Coin Toss

Suppose a gambler is playing a game of coin toss. The gambler tosses a coin in the air and the coin can land on either side. The gambler has a probability p of winning the bet and a probability $q = 1 - p$ of losing the bet. We assume that the game is favorable to the gambler so that the coin is not a fair coin and the odds of winning are in the gambler's favor $p - q > 0$. Additionally, we are looking at a series of coin tosses and the probability of winning does not ever change between bets.

X_t is defined as the amount of capital a gambler holds after a series of t bets. The gambler begins a series of N independent bets with an initial amount of capital X_0, at time $t = 0$, and for every bet the gambler bets a fraction of capital defined as $f \in [0, 1]$.

If the gambler wins the bet then his capital increases by $X_0 \cdot f$ and if the gambler loses the bet then his capital decreases by $X_0 \cdot f$. This type of game is called a binomial game because the outcome of the bet is a discrete set of outcomes, either the gambler wins a certain amount or he loses a certain amount. For example if the gambler plays a single bet and wins then his capital is now

$$X_1 = X_0 + X_0 \cdot f = X_0 \cdot (1 + f)$$

and if he loses the single bet his capital is

$$X_1 = X_0 - X_0 \cdot f = X_0 \cdot (1 - f)$$

Assume he played two bets, he won the first and lost the second. His capital is now

$$X_2 = X_1 \cdot (1 - f) = X_0 \cdot (1 + f) \cdot (1 - f)$$

This type of bet has odds of 1 to 1 because the amount of capital the gambler wins or loses every bet is only the fraction of capital he bet. The odds never change during the series of bets played so this type of betting system is called fixed odds betting. The fraction that is bet on every individual coin toss also never changes and is called a fixed fraction bet.

The gambler is playing a series of $N \in \mathbb{N}$ bets with $W_N \in \mathbb{N}$ the number of wins and $L_N \in \mathbb{N}$ the number of losses and $W_N + L_N = N$. Using the equation for X_2 above, we can see in a similar manner that if the gambler won W_N bets and lost L_N bets then his current capital after N bets is

$$X_N = X_0 \cdot (1 + f)^{W_N} \cdot (1 - f)^{L_N} \qquad \text{(eq 2.1)}$$

Now the gambler needs to decide what is the best f so he can maximize his capital growth. To find the optimal f, which we call f^*, we will first show a naive approach of determining f^* by maximizing the expected value of (eq 2.1).

2.1.1 Maximizing the expected value

W_N is the amount of wins the gambler has in a series of N bets so W_N is a binomial random variable with N bets and p probability of winning, $W_N \sim B(N,p)$. L_N is the amount of losses the gambler has in a series of N bets so $L_N = N - W_N$ and L_N is a random variable as well so $L_N \sim B(N, 1-p)$. Because W_N and L_N are used in (eq 2.1) then X_N is a random variable as well. Obviously, one of the objectives of the gambler is to maximize his capital at the end of the series of bets by using f^* for every bet and the classic way to accomplish that would be to maximize the expected value of the random variable X_N and use that to get f^*.
As a reminder, the formal definition of the probability distribution of a binomial random variable with N trials and p probability $X \sim B(N,p)$ is

$$P(X=k) = \binom{N}{k} \cdot p^k \cdot (1-p)^{N-k}.$$

This is the probability of getting exactly k successes in N trials. The expected value of a random variable X with N possible values is

$$E[X] = \textstyle\sum_{x=0}^{N} P(X=x) \cdot x.$$

Using the formal definition, the expected value of the random variable X_N is

$$E[X_N] = \textstyle\sum_{w=0}^{N} X_0 \cdot P(W_N = w) \cdot (1+f)^w \cdot (1-f)^{N-w}.$$

Replace the probability function $P(W_N = w)$ with the definition stated above

$$\begin{aligned} E[X_N] &= X_0 \cdot \textstyle\sum_{w=0}^{N} \binom{N}{w} p^w \cdot q^{N-w} \cdot (1+f)^w \cdot (1-f)^{N-w} \\ &= X_0 \cdot \textstyle\sum_{w=0}^{N} \binom{N}{w} \cdot (p \cdot (1+f))^w \cdot (q \cdot (1-f))^{N-w}, \end{aligned}$$

and using the binomial formula

$$(a+b)^N = \textstyle\sum_{m=0}^{N} \binom{N}{m} \cdot a^m \cdot b^{N-m},$$

we can simplify the expression to

$$\begin{aligned} E[X_N] &= X_0 \cdot (p \cdot (1+f) + q \cdot (1-f))^N \\ &= X_0 \cdot (f \cdot (2 \cdot p - 1) + 1)^N \qquad \text{(eq 2.2)} \end{aligned}$$

(eq 2.2) is a monotonically increasing function if $p > \frac{1}{2}$

$$p > \tfrac{1}{2} \rightarrow f \cdot (2 \cdot p - 1) + 1 > f \cdot (2 \cdot \tfrac{1}{2} - 1) + 1 = 1$$

We can clearly see above that $E[X_N]$ is a monotonically increasing function if $p > \frac{1}{2}$ and a monotonically decreasing function if $p < \frac{1}{2}$. Using the monotonicity and the constraint $p > \frac{1}{2}$, the maximum of the function is at the edge of the interval $f \in [0,1]$ where $f = 1$. The value of $E[X_N]$ at the point $f = 1$ is

$E[X_N] = X_0 \cdot (2 \cdot p)^N$. Because the maximum is at the point $f = 1$ the gambler should always bet everything he has on every bet in the series. Obviously if he bets everything he has at each bet and the gambler wins every bet then his capital will increase the most. The downside is that if he loses just one bet then he loses everything. If the constraint on the probability of winning is $p < \frac{1}{2}$ then the maximum of the function is at the point $f = 0$ which basically tells the gambler not to bet. Therefore, the classic approach to calculating f^* is not applicable if we want to lower the risk of catastrophic failure, therefore another way is needed to calculate f^*. Next we will be demonstrating the Kelly way of calculating f^* which is done by maximizing the utility function.

2.1.2 Maximizing the utility function

According to Kelly, in order to determine the f^*, what is first needed is to define the utility function $g(f)$ for the game the gambler is playing and then maximize it to get f^*. The utility function must have the following properties:
• Non-linear - The utility function must be non-linear because it must have stationary points where the marginal utility becomes zero.
• Continuous - The utility function must be defined in the interval $f \in (0, 1)$ because the gambler cannot bet less then zero capital and cannot bet more then all his capital.
Now we will define the utility function of the coin toss game. First α is defined as the geometric capital growth rate of a N series of bets

$$\alpha_N = \sqrt[N]{\frac{X_N}{X_0}},\ \alpha = lim_{N \to \infty} \alpha_N$$

the natural log is applied to both sides to give

$$\ln(\alpha_N) = \frac{1}{N} \cdot \ln\left(\frac{X_N}{X_0}\right) \qquad \text{(eq 2.3)}$$

Using (eq 2.1), divide by X_0

$$\tfrac{X_N}{X_0} = (1 + f)^{W_N} \cdot (1 - f)^{L_N}$$

the natural log is applied to both sides

$$G(f) = \ln\left(\frac{X_N}{X_0}\right) = W_N \cdot \ln(1 + f) + L_N \cdot \ln(1 - f) \qquad \text{(eq 2.4)}$$

G(f) is the logarithmic capital growth function of N bets. Because what is interesting is the case where N is a large number, the utility function after a long series of bets is needed. Divide $G(f)$ by N

$$\tfrac{1}{N} G(f) = \tfrac{1}{N} \cdot \ln(\tfrac{X_N}{X_0}) = \tfrac{W_N}{N} \cdot \ln(1 + f) + \tfrac{L_N}{N} \cdot \ln(1 - f)$$

now using (eq 2.3)

$$\ln(\alpha_N) = \frac{1}{N}G(f) = \frac{1}{N} \cdot \ln(\frac{X_N}{X_0}) = \frac{W_N}{N} \cdot \ln(1+f) + \frac{L_N}{N} \cdot \ln(1-f)$$

and with a very large N and applying the Law of Large Numbers (The law describes the result of performing the same experiment a large number of times with the average result being close to the expected value)

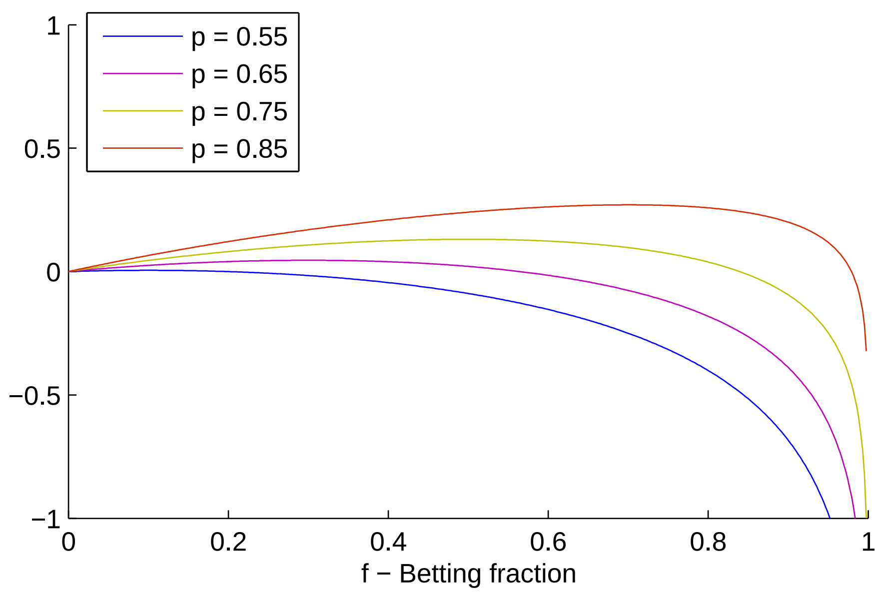
$$g(f) = \ln(\alpha) = lim_{N\to\infty} \ln(\alpha_N) = lim_{N\to\infty} \frac{1}{N} \ln\left(\frac{X_N}{X_0}\right) =$$

$$lim_{N\to\infty} \frac{W_N}{N} \cdot \ln(1+f) + lim_{N\to\infty} \frac{L_N}{N} \cdot \ln(1-f)$$

$$= p \cdot \ln(1+f) + q \cdot \ln(1-f) \qquad \text{(eq 2.5)}$$

The function $g(f)$ is the utility function of the capital after a very large series of bets. Figure 1 shows the graph of $g(f)$ using different values of p: 0.55, 0.65, 0.75, 0.85, respectively.

Figure 1: graph of g(f)



It can be seen that $g(f)$ is non-linear and continuous in the interval $(0, 1)$. Now using $g(f)$ its possible to get the optimal betting ratio, f^*.

$$g'(f) = \frac{p}{1+f} - \frac{q}{1-f} = 0 \longrightarrow$$

$$p - q = f(p+q) \longleftrightarrow p - q = f(p + (1-p))$$

$$f^* = p - q = 2p - 1 \qquad \text{(eq 2.6)}$$

(eq 2.6) is the critical point of the function. The next step is to prove that f^* is the maximum of $g(f)$ in the interval $f \in (0,1)$. The second derivative of $g(f)$ is

$$g''(f) = \frac{-p}{(f+1)^2} - \frac{q}{(f-1)^2}$$

$g''(f)$ is always negative for any f therefore f^* is a local maximum of the function $g(f)$. On the boundaries the value of the utility function is: $g(f=0) = 0$ and on $lim_{f \to 1} g(f) = -\infty$ (We can easily see this in figure 1). $g(f)$ is continuous in the interval $[0,1)$ therefore the utility function has a maximum value between the boundaries of the interval. The value of the utility function on the local maximum is $g(f = f^*) = \ln(2) + p \cdot \ln(p) + q \cdot \ln(q)$. Therefore the global maximum is:

- If $p \leq \frac{1}{2}$ then the global maximum is $f = 0$. This is because if $p \leq \frac{1}{2}$ then $f^* = 2 \cdot p - 1 \leq 0$.

- If $p > \frac{1}{2}$ then the global maximum is $f = f^*$ ($f^* \in (0,1)$).

Another point of interest is the maximum geometric capital growth rate when using f^*. Using (eq 2.5), α is

$$\alpha = \exp(p \cdot \ln(1+f) + q \cdot \ln(1-f))$$

and after inserting f^* into the equation

$$\alpha^* = \exp(p \cdot \ln p + q \cdot \ln q + \ln 2) = 2 \cdot p^p \cdot q^q \qquad \text{(eq 2.7)}$$

This equation provides the maximum geometric capital growth rate when betting using f^*.
To summarize, when playing a series of coin tosses where the odds are in the gambler's favor, we presented two different ways to determine how much to bet on a single coin toss. The gambler can either bet everything he has on each single coin toss and risk losing all his capital if he loses just once, or he can use the Kelly Criterion and get the optimal fraction of capital to bet at every bet in the series which helps the gambler lower his risk of catastrophic failure because he is not betting everything he has, just a fraction of it, so he will always have a certain amount of capital at the end of the series. Using the Kelly Criterion and not maximizing the expected value will lower his short term gain but increase his long term gain because he will most likely increase his capital rather then decrease it if he plays a large number of games because of the edge he has in the game.

After analyzing the specific case of a game with odds 1 to 1, a more general analysis can be made of the case of a game with odds V to 1 where $V \in [1, \infty)$, the difference being that if the gambler wins he gets V times his bet and if he loses then he loses just his bet. For this case, (eq 2.1) is replaced by

$$X_N = X_0 \cdot (1 + V \cdot f)^{W_N} \cdot (1-f)^{L_N} \qquad \text{(eq 2.8)}$$

Before defining the utility function we need to know for what values of p is it worth the gambler to make a bet in this type of game. Using (eq 2.2) and modifying it for the more general case of a game of V to 1 odds the expected value of the capital the gambler holds after N series of games is

$$E[X_N] = X_0 \cdot (p \cdot (1 + V \cdot f) + q \cdot (1 - f))^N \ .$$

We assume $N = 1$ and check when the expected value is larger then the initial capital

$$E[X_1] = X_0 \cdot (p \cdot (1 + V \cdot f) + q \cdot (1 - f)) > X_0$$

$$\leftrightarrow X_0 \cdot (p + p \cdot V \cdot f + q - q \cdot f) > X_0$$

$$\leftrightarrow X_0 \cdot (p + q) + X_0 \cdot f \cdot (p \cdot V - q) > X_0$$

$$\leftrightarrow X_0 + X_0 \cdot f \cdot (p \cdot V - q) > X_0$$

$$\leftrightarrow 1 + f \cdot (p \cdot V - q) > 1$$

$$\leftrightarrow f \cdot (p \cdot V - q) > 0$$

f is in the interval $(0, 1)$ therefore

$$(p \cdot V - q) > 0$$

After isolating p we get

$$p > \frac{1}{1 + V} \leftrightarrow E[X_1] > X_0 \ .$$

This means that for any value of $1 > p > \frac{1}{1+V}$ the expected value is positive. If $0 < p < \frac{1}{1+V}$ the expected value is negative meaning the gambler is losing capital and therefore the gambler should not place a bet in the game. This generalizes the assumption $p > 0.5$ made in the game of 1 to 1 odds.

The next step is to define the utility function. For this type of game it is slightly different then before

$$g(f) = lim_{N \to \infty} \ln(\alpha) = lim_{N \to \infty} \frac{1}{N} \cdot \ln\left(\frac{X_N}{X_0}\right)$$

$$= p \cdot \ln(1 + V \cdot f) + q \cdot \ln(1 - f) \qquad \text{(eq 2.9)}$$

Figures 2 and 3 represent two graphs of $g(f)$ for the general coin toss with values of V: 10, 100, and the same values of p as shown in the previous figure.

Figure 2: graph of g(f), V = 10

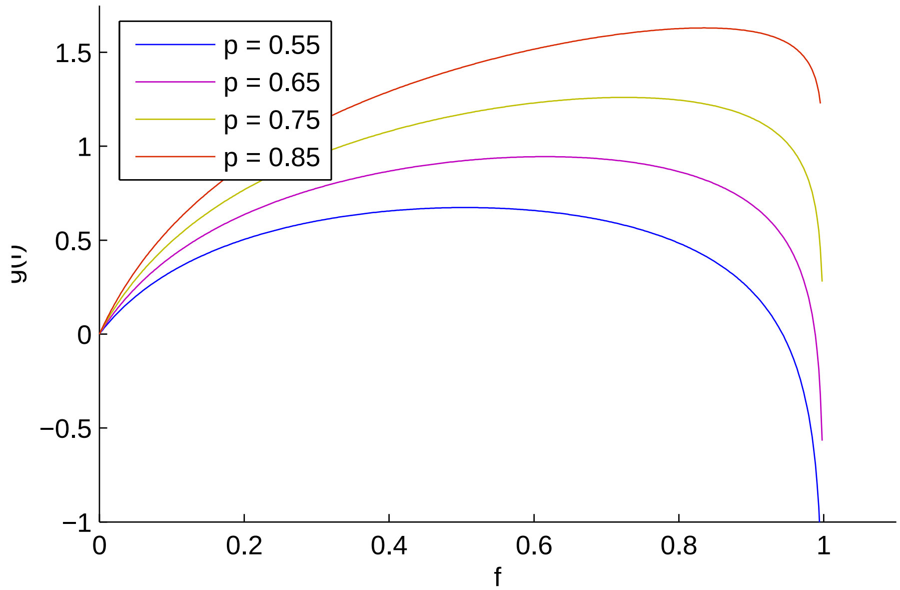
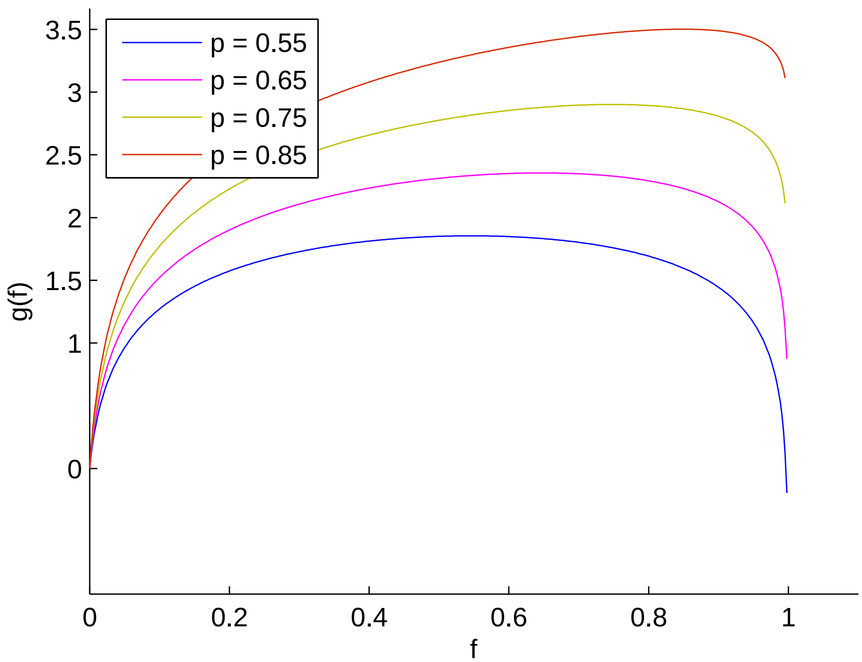


Figure 3: graph of g(f), V = 100



Again it can be seen that $g(f)$ is continuous in the interval $(0, 1)$. To compute f^* we use the derivative of $g(f)$

$$g'(f) = \frac{Vp}{1 + V \cdot f} - \frac{q}{1 - f} = 0$$

$$\leftrightarrow f^* = \frac{p \cdot (V + 1) - 1}{V} \qquad \text{(eq 2.10)}$$

As before, the next step is to show that f^* is the maximum of $g(f)$ in the interval $[0, 1)$. $g(f)$ is a continuous function in the interval $(0, 1)$ therefore it must have a maximum in the interval or on the edges of the interval. The second derivative of $g(f)$ is

$$g''(f) = \frac{-q}{(f-1)^2} + \frac{-V^2 \cdot p}{(V \cdot f + 1)^2} < 0$$

therefore f^* is a maximum of $g(f)$. The next step is to prove that $f^* \in [0, 1]$. By inserting the constraint $p > \frac{1}{1+V}$ and $p < 1$ into (eq 2.10) we get

$$f^* > \frac{\frac{1}{V+1} \cdot (V + 1) - 1}{V} = 0$$

$$f^* < \frac{1 \cdot (V + 1) - 1}{V} = 1 \ .$$

It has been proven that $f^* \in [0,1]$. At the edges of the interval $f \in [0,1)$ the value of $g(f)$ is: $g(0) = 0$, $lim_{f\to 1} g(f) = -\infty$ therefore f^* is the maximum of $g(f)$ in the interval.

Now it is possible to get the maximum geometric capital growth rate when using f^* in a game of odds V to 1.

$$\alpha = \exp(p \cdot \ln(1 + V \cdot f) + q \cdot \ln(1 - f)) = (1 + V \cdot f)^p \cdot (1 - f)^q$$

$$\alpha^* = (p \cdot (V + 1))^p \cdot \left(\frac{V - p \cdot (V + 1) + 1}{V} \right)^q \qquad \text{(eq 2.11)}$$

the equation can be simplified

$$\alpha^* = (p \cdot (V + 1))^p \cdot \left(\frac{q \cdot (V + 1)}{V} \right)^q \qquad \text{(eq 2.12)}$$

We can summarize the coin toss game using the following points

- Betting all the capital in the gambler's possession while playing a large series of bets, $f = 1$, is a risky strategy which will most likely lead to the gambler losing everything.

- Using the Kelly approach by defining a utility or growth function and maximizing the capital growth rate will give the gambler an optimal f which will bring his risk of losing all his capital to zero since he is always betting a fraction of capital and not the whole amount he holds.

- If the game is not favorable, then Kelly tells the gambler not to bet. Equations 2.7 and 2.11 cannot work if the gambler does not have an edge so they would become negative, meaning the maximum of the utilty function is at $f = 0$, which is basically saying do not bet.

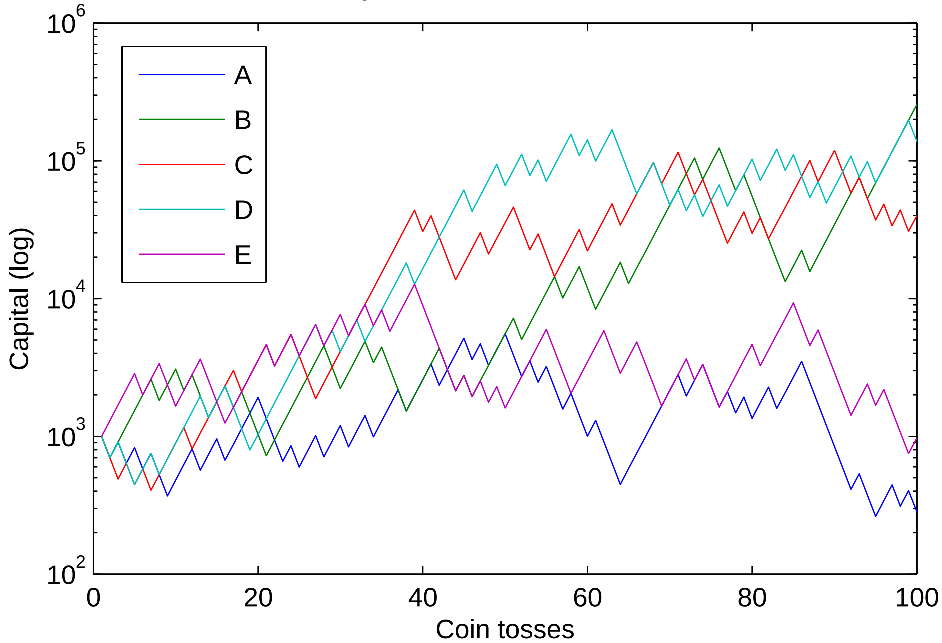
- What happens if the gambler chooses to use a fraction of capital $f_C < f^*$ or $f_C > f^*$? If the gambler chooses the former then because f^* is a maximum of the function $g(f)$ the gambler's capital will continue to increase but at a slower rate then if he had used f^*. This is because the function $g(f)$ is positive in the interval $(0, f^*)$. If the gambler chooses the latter then his capital can either grow at a slower rate or even decrease over time. We will see a simulation of this case in the following section. Additionally, this conclusion can be described differently saying: if the utility function is positive then the gamblers capital will increase; $g(f) > 0$ then $lim_{N\to\infty} X_N = \infty$ almost surely after a long series of bets. On the other hand, if $g(f) < 0$ then $lim_{N\to\infty} X_N = 0$ almost surely.

2.1.3 Coin Toss Simulations

This section contains simulations that describe various scenarios of the Coin Toss game. A gambler is playing a single game of coin toss with a biased coin where he has a probability of p = 0.65 of winning with odds of 1 to 1. He starts out with an initial capital $X_0 = 1000$ dollars. If using the Kelly Criterion then by utilizing (eq 2.6) for choosing an optimal betting ratio the gambler will bet $f^* = 2 \cdot 0.65 - 1 = 0.3$. According to Kelly the gambler should bet 30 percent of his current capital every coin toss to make his capital increase the fastest with zero risk. The utility function at the maximum is $g(f^*) = 0.0457$ and using (eq 2.7) the geometric capital growth rate between bets is 1.0467 or 4.6 percent geometric growth rate.

Figure 4 contains five simulations of the gambler playing $N = 100$ coin tosses with the data mentioned in the game above and shows the amount of capital the gambler holds after every bet.

Figure 4: Example 1



It can be seen that in simulation D that at the end of the series of coin tosses the gambler's capital reaches a factor of almost 10^2 of his initial capital and for simulations B and C the gambler's capital reaches a peak of almost $10^{4.5}$. For simulations A and E the gambler experiences a loss of capital and at the end of the series is below his initial capital of 10^3. This shows us the random variation between different series of coin tosses where the gambler can experience losses in some simulations and high earnings in other simulations.
The next three figures are simulations of the gambler playing N coin tosses with the same data mentioned above and shows the amount of capital the gambler holds after every bet. Figure 5 contains five simulations of $N = 100$ coin tosses

using the betting ratio of $f_1 = 1$. Figure 6 contains five simulations of $N = 100$ coin tosses using the optimal betting fraction $f_2 = 0.3$. In addition, Figure 7 is a logarithmic graph of the second figure.

Figure 5: Example 2

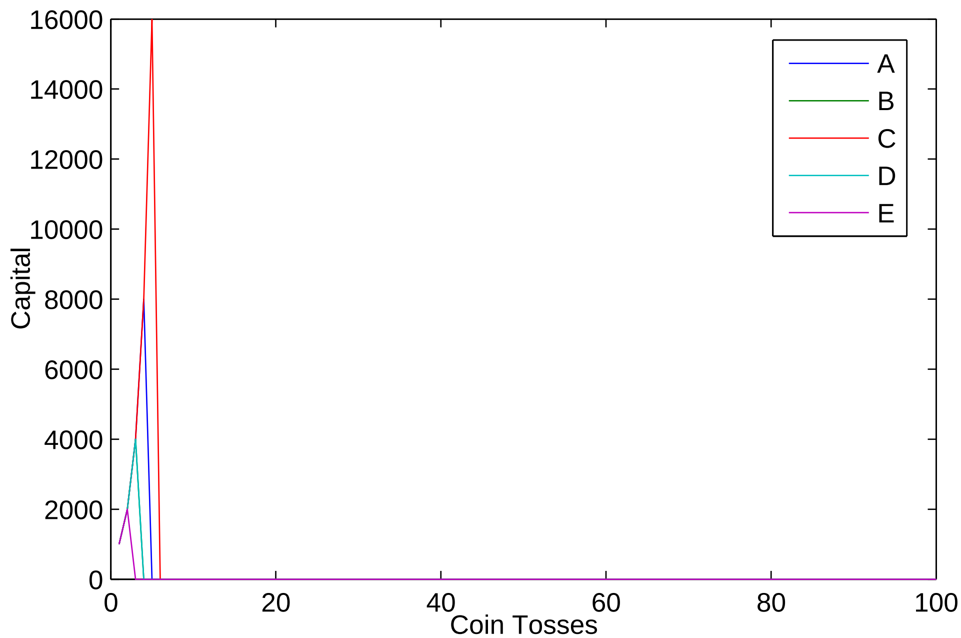


Figure 6: Example 3

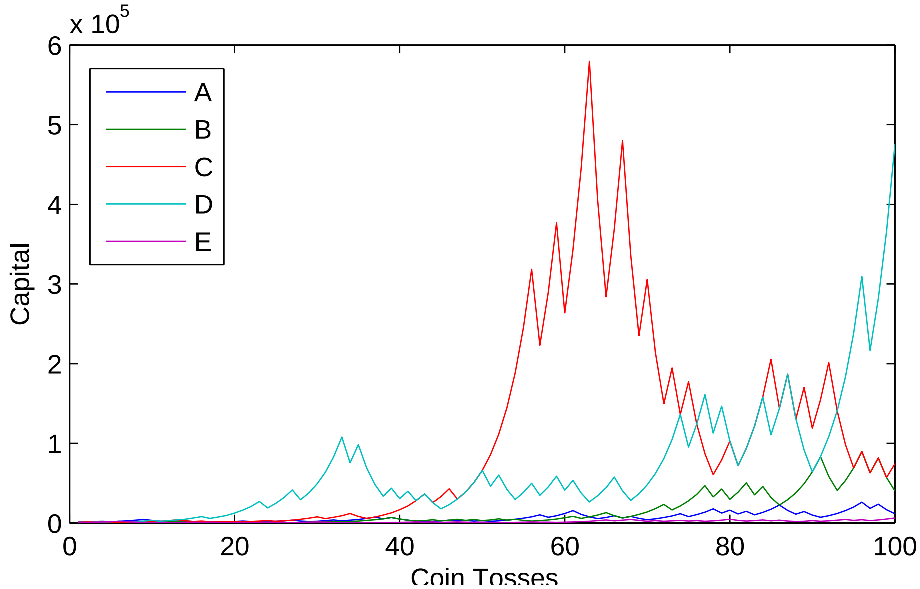
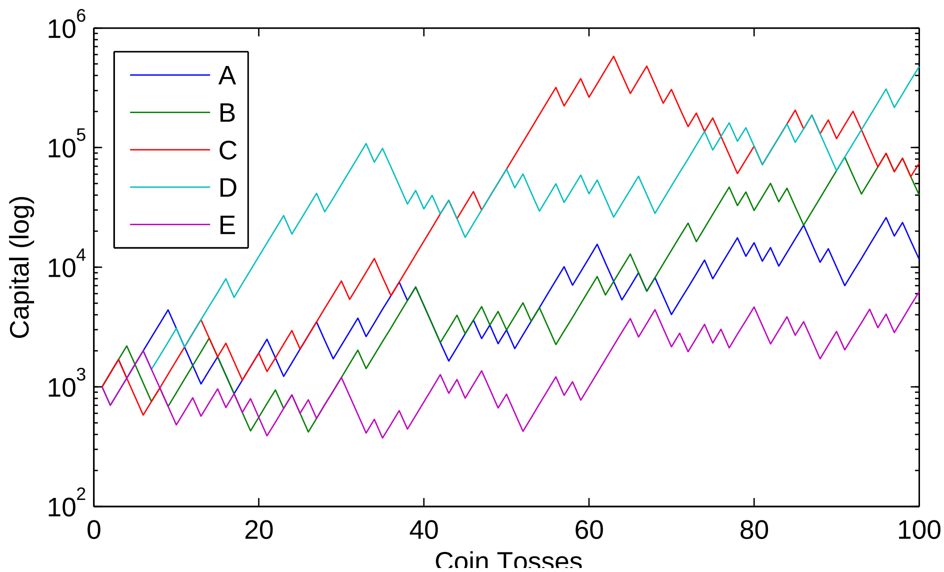


Figure 7: Example 4

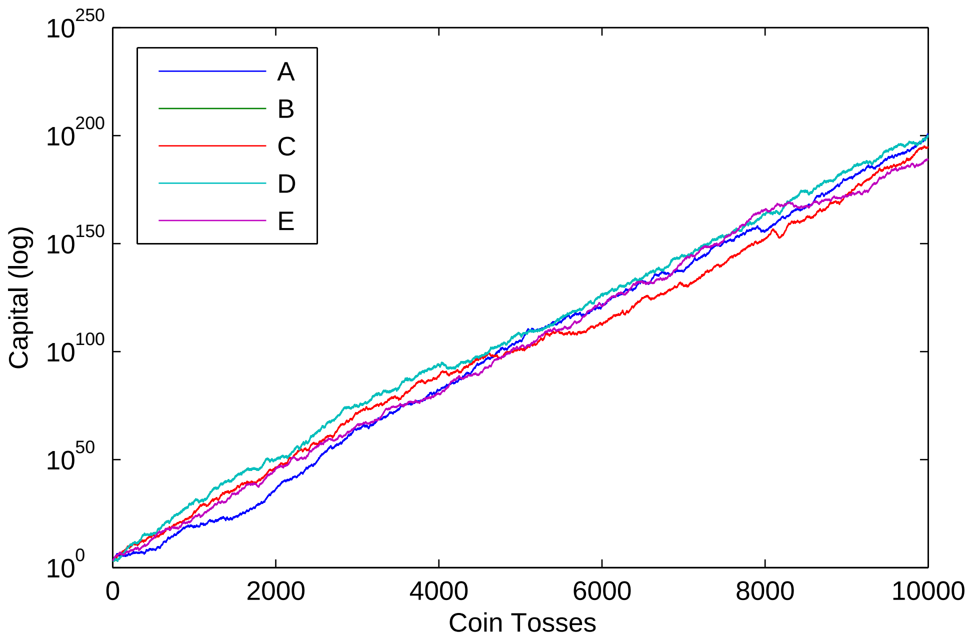


In Figure 5 it is noticeable is that if the gambler uses the strategy of $f_1 = 1$ then he will lose all his capital after just a few coin tosses. Using the strategy of

$f_2 = 0.3$ in Figures 6 and 7 has increased the gamblers capital and has brought his risk of catastrophic failure to zero because he bets a portion of his capital every coin toss.

Figure 8 is a simulation of a gambler playing a game of $N = 10000$ coin tosses with the same data mentioned above ($f = 0.3$). The figure contains five curves using the Kelly Criterion strategy.

Figure 8: Example 5



From this figure the geometric capital growth rate can be calculated and can be compared to see if its close to the 4.6 percent geometric growth rate calculated using (eq 2.7). We can calculate the geometric capital growth rate using (eq 2.3). For example, on curve A the gambler's capital after N = 10000 coin tosses is $5.6274 \cdot 10^{200}$

$$\ln(\alpha_A) = \frac{1}{10000} \cdot \ln\left(\frac{5.6274 \cdot 10^{200}}{10000}\right) = 0.0453$$

On curve D the gambler's capital after $N = 10000$ coin tosses is $1.9302 \cdot 10^{200}$.

$$\ln(\alpha_D) = \frac{1}{1000} \cdot \ln\left(\frac{1.9302 \cdot 10^{200}}{1000}\right) = 0.0452$$

The geometric capital growth rate of curve A is $\alpha_A = \exp(0.0453) = 1.0463$ which is a 4.63 percent capital growth rate. The geometric capital growth rate of curve D is $\alpha_D = \exp(0.0452) = 1.0462$ which is a 4.62 percent capital growth rate. Both growth rates are close to the 4.6 percent calculated previously. For smaller simulations we will get a disparity between the calculated growth rate and the actual growth rate of the simulation.

Another point of interest is what happens if the gambler uses the Kelly Criterion for determining f^* but bets f_C which is smaller or bigger then f^*? Using the same data mentioned in the initial example, if the gambler uses $f_C = 0.1 < f^* = 0.3$ then $g(f_C) = 0.0250$ and $\alpha = 1.02$ meaning his capital growth rate is at around 2 percent which is a lower growth rate then if he had used f^*. If the gambler uses $f_C = 0.5 > f^* = 0.3$ then $g(f_C) = 0.0209$ and $\alpha = 1.03$ which is a growth rate of 3 percent meaning he will again still experience slower capital growth.

Figures 9, 10 and 11 each contain a simulation of the gambler playing $N = 1000$ coin tosses again with the same data mentioned in the initial example and shows the amount of capital the gambler holds after every bet. Each Figure contains three curves: A, B and C, where each curve uses a different f: $f_1 = 0.1 < f^*$, $f^* = 0.3$ and $f_2 = 0.5 > f^*$ respectively.

Figure 9: Example 6

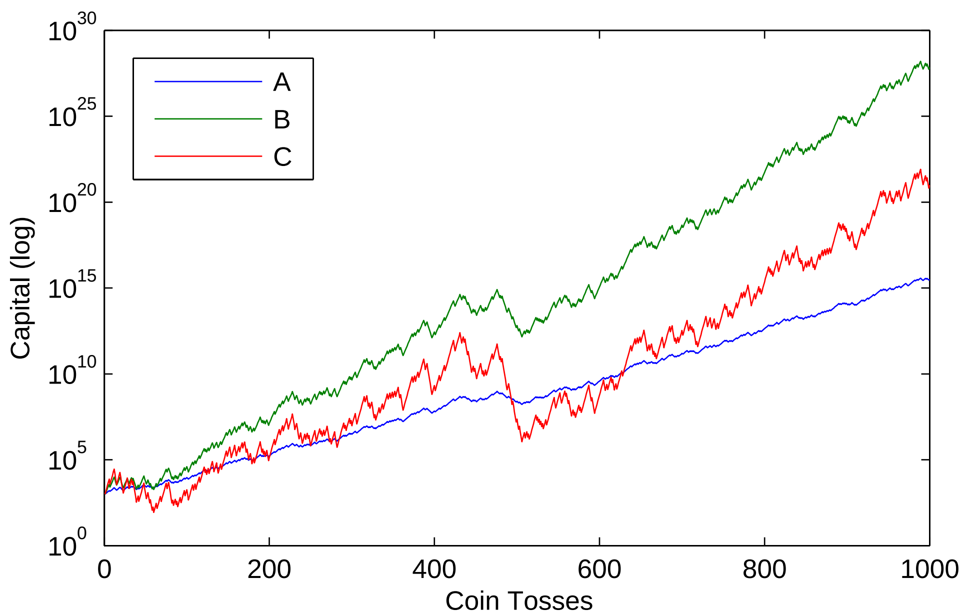


Figure 10: Example 7

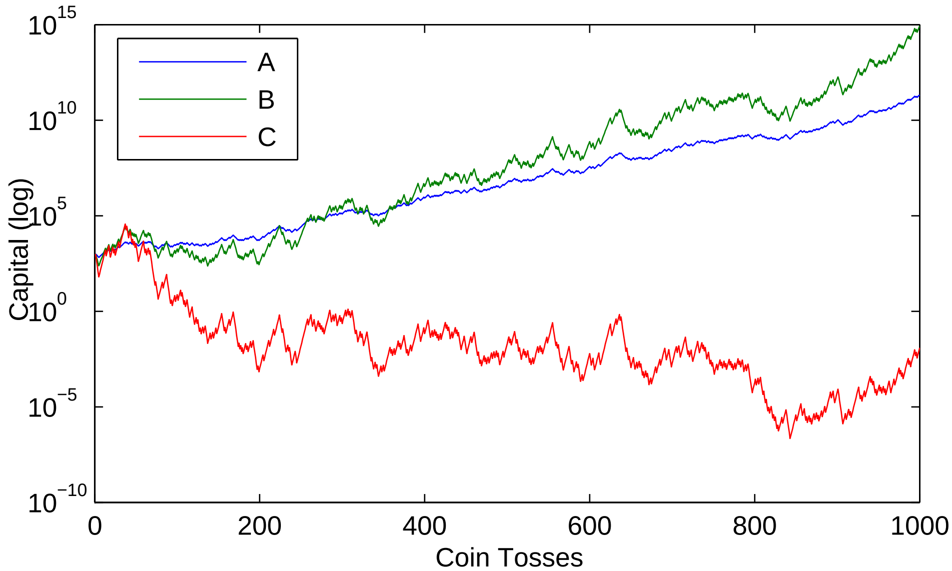
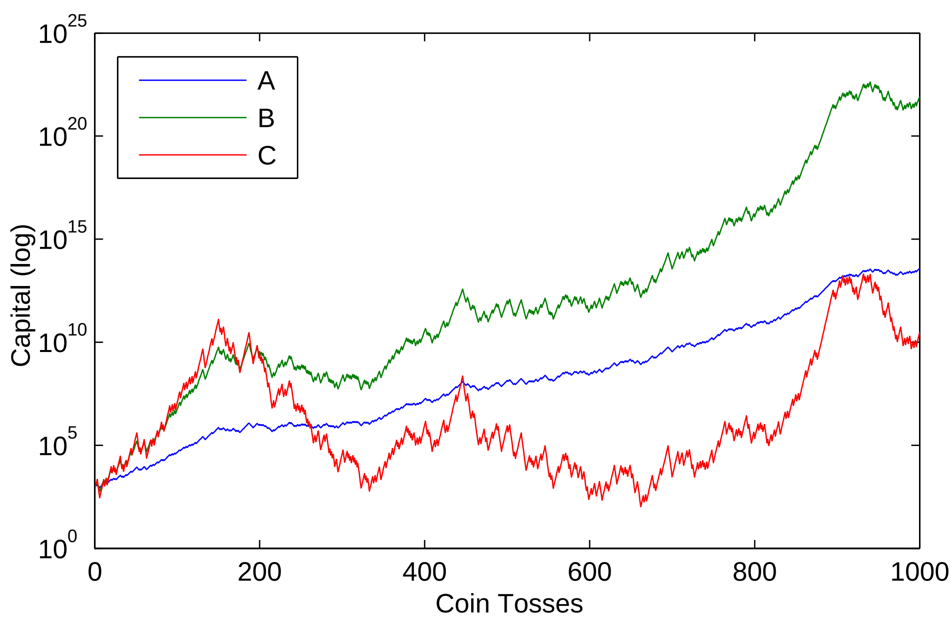


Figure 11: Example 8



What is noticeable is that the best strategy for the gambler to use would be

f^*, his capital peaks the highest out of all the other strategies in all the figures. When using $f_2 > f^*$ the gamblers capital grows, but not as well as it could had the gambler used f^*. But its still very much possible for the gambler to peak with a larger amount of capital using f_2 when playing short term games, below 250 coin tosses, then he would peak using f^*, an example will be shown in the following figures. But when using f_2 his risk of losing a large amount of money gets higher the more f_2 is above f^*. This can be seen in figures 10 and 11 where the gambler experiences a large drop in capital because of a few coin toss losses. They clearly shows the additional risk the gambler takes when using f_2 and the less risk he takes when using f^*. This tells us that there is a trade-off between a higher return if the gambler wins and the amount of risk he is taking so if the gambler uses f^* he gets a certain balance of taking a bet with a fair amount of risk and also getting a good return of his bet if he wins.

Figure 12: Example 9, Sample point: $N = 25, f_1 = 0.1, f^* = 0.3, f_2 = 0.5$

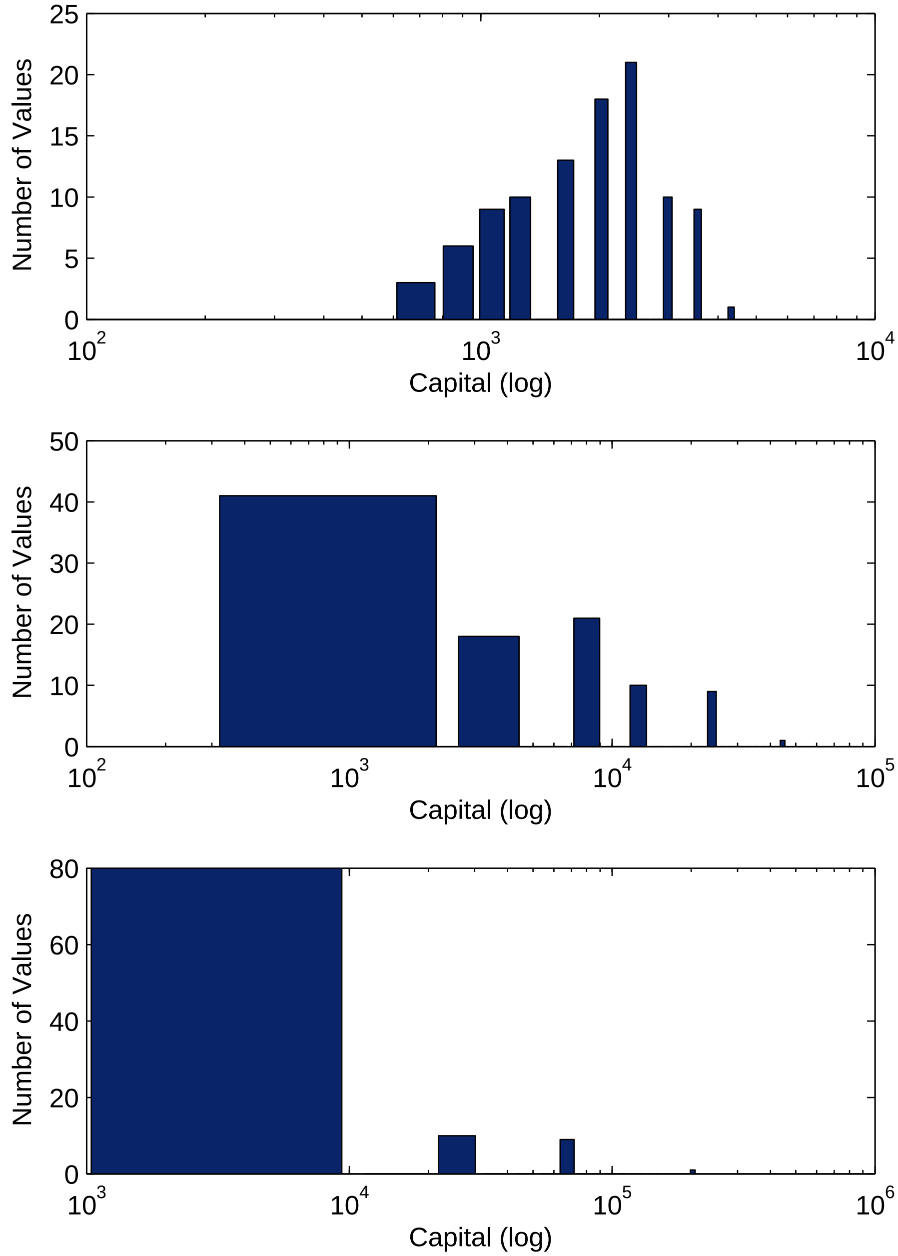


Figure 13: Example 10, Sample point: $N = 250, f_1 = 0.1, f^* = 0.3, f_2 = 0.5$

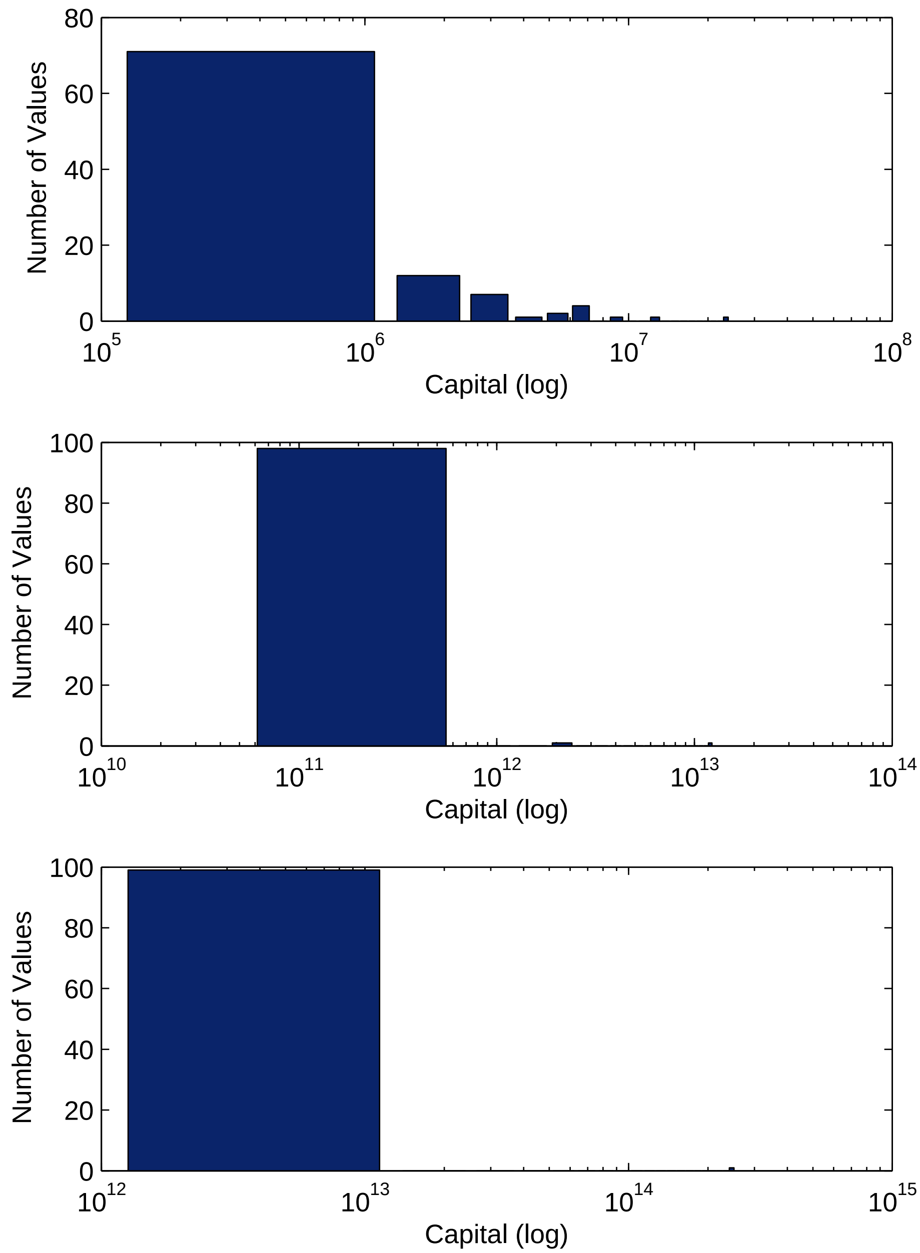
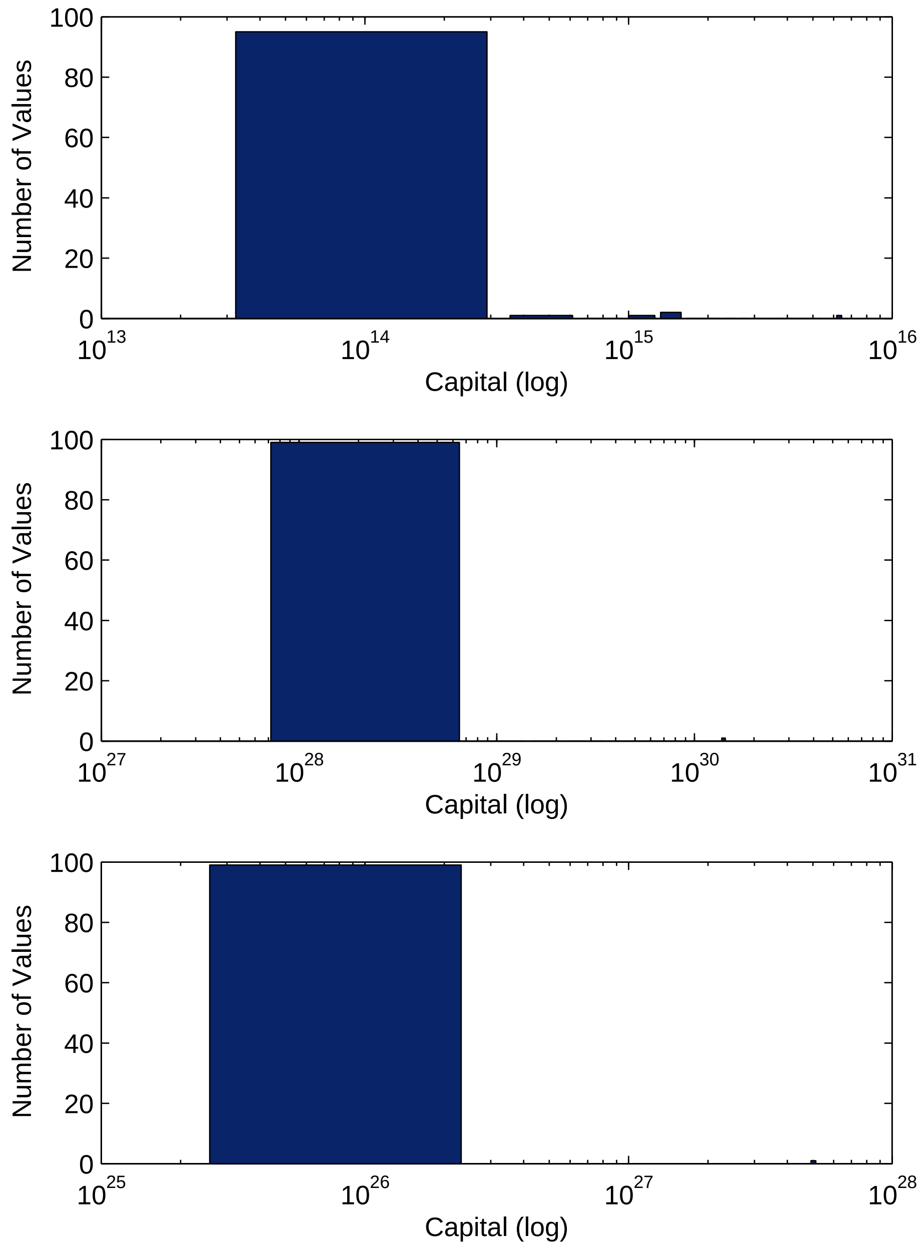


Figure 14: Example 11, Sample point: $N = 900, f_1 = 0.1, f^* = 0.3, f_2 = 0.5$



Figures 12, 13 and 14 are histograms containing $H = 100$ simulations of a gambler playing $N = 25, 250, 900$ coin tosses with the data used in figures 9, 10 and 11.

• In a histogram figure the X axis is intervals of capital at the end of N amount of games and the Y axis is, at the end of a simulation, how many simulations the gambler had an amount of capital that falls into a specific interval. The simulations in each histogram end after reaching a specific number of coin tosses.

• Histogram figure 12 shows the amount of capital the gambler had after exactly 25 coin tosses.

• Histogram figure 13 shows the amount of capital the gambler had after exactly 250 coin tosses.

• Histogram figure 14 shows the amount of capital the gambler had after exactly 900 coin tosses.

We can see in figure 12 that using either strategy of f_2 or f^* would be a good strategy for the gambler because they both yield higher earnings then using strategy f_1 for a short number of coin tosses. In figure 13 we can see again that using f_2 increases the gambler's capital and f^* also increases his capital, although in most of the simulations f_2 increases the gamblers capital more then any other strategy. In figure 14 we can clearly see that using f^* would be the best choice of strategy for the gambler because in all the simulations using f^* the gambler's capital increases the most then using f_2. Another point of interest is when using $f_1 < f^*$ the gamblers capital grows at a steady but very slow growth rate as compared to the other two betting fractions and that is why using f_1 is the worst strategy for the gambler.

To summarize these examples, the conclusion is that for the coin toss the Kelly Criterion provides very easy equations for finding the f^*, can lead to a high increase in the gamblers capital when playing a large amount of bets and lowers the risk of losing all the capital. Another conclusion is that the gambler can choose an f that is close to f^* and have his capital grow in the long term, but at a much slower growth rate then if he had used f^*. The next section discusses different variations of the original coin toss game and how they apply to various games in Sports Betting.

3 Sports Betting

Sports betting is the activity of predicting sports results and placing a wager on the outcome and can include games such as Horse races and Blackjack. Just as in the case of the coin toss, it's possible to use the Kelly Criterion to make optimal bets in sports betting. The game discussed in this paper is the game of Horse Racing.

3.1 Horse Racing

In Horse Racing a gambler can make several different types of bets. He can bet which horse will win the race, he can bet which horse will finish in second or third place or he can make several different bets on several different horses simultaneously. There are many other types of bets a gambler can make but the majority are variations of the betting types listed above. What is common among all the different types of bets is that a gambler has to choose on which horse/s to place his bet/s where each horse has a different probability of finishing in a certain place. In this section we will analyze the case of a gambler betting simultaneously on all the horses in a race. The gambler will divide all his capital on bets on all the horses and a bet on each horse is a bet that the horse will win the race.

Suppose a gambler wants to divide all his capital on bets on three different horses, A, B and C winning a horse race. Each horse has a different probability of winning the race, $p_A, p_B, p_C \in [0,1]$, and the gambler knows the probabilities of success of each horse. Also, the fraction of capital he bets on each horse is $f_A, f_B, f_C \in [0,1], f_A + f_B + f_C = 1$. The probabilities of each horse winning do not change between races. The odds of the game are 1 to 1. For example, if the gambler plays a single race and horse A wins the race, then of course horses B and C lose and the gambler's capital is now

$$X_1 = X_0 \cdot (1 + f_A - f_B - f_C)$$

The gambler gains the amount of capital he bet on horse A and loses the amount of capital he bet on horses B and C. Next, assume he bets on two consecutive races, in the first race horse A won and in the second race horse C won, capital is now

$$X_2 = X_1 \cdot (1 - f_A - f_B + f_C) = X_0 \cdot (1 + f_A - f_B - f_C) \cdot (1 - f_A - f_B + f_C)$$

The gambler is playing a series of $N \in \mathbb{N}$ races with

- ${W_N}^A \in \mathbb{N}$ - amount of wins for horse A,
- ${W_N}^B \in \mathbb{N}$ - amount of wins for horse B,
- ${W_N}^C \in \mathbb{N}$ - amount of wins for horse C,

and ${W_N}^A + {W_N}^B + {W_N}^C = N$. Using the equation of X_2 above, we can see in a similar manner the gambler's current capital after N bets is

$$X_N = X_0 \cdot (1 + f_A - f_B - f_C)^{{W_N}^A} \cdot (1 - f_A + f_B - f_C)^{{W_N}^B} \cdot (1 - f_A - f_B + f_C)^{{W_N}^C} \qquad \text{(eq 3.1)}$$

Now the gambler needs to decide how to optimally divide his capital for the three bets. To accomplish this under Kelly, a utility function must first be defined. Just as in the previous case of the coin toss, divide by X_0 and apply the natural log.

$$G(f_A, f_B, f_C) = \ln(\tfrac{X_N}{X_0}) = {W_N}^A \cdot \ln(1 + f_A - f_B - f_C) +$$

$${W_N}^B \cdot \ln(1 - f_A + f_B - f_C) + {W_N}^C \cdot \ln(1 - f_A - f_B + f_C)$$

$G(f)$ is defined as the logarithmic capital growth function of a series of bets. Dividing G(f) by N and using (eq 2.3)

$$\ln(\alpha_N) = \tfrac{1}{N} G(f) = \tfrac{1}{N} \cdot \ln(\tfrac{X_N}{X_0}) = \tfrac{{W_N}^A}{N} \cdot \ln(1 + f_A - f_B - f_C) + \tfrac{{W_N}^B}{N}$$

$$\cdot \ln(1 - f_A + f_B - f_C) + \tfrac{{W_N}^C}{N} \cdot \ln(1 - f_A - f_B + f_C)$$

and with a very large N and applying the Law of Large Numbers

$$g(f_A, f_B, f_C) = \ln(\alpha_N) = p_A \cdot \ln(1 + f_A - f_B - f_C) + p_B \cdot \ln(1 - f_A + f_B - f_C) +$$
$$p_C \cdot \ln(1 - f_A - f_B + f_C). \qquad \text{(eq 3.2)}$$

This equation is the utility function of the capital in a horse race after a very large series of bets. The next step is to find the maximum. This problem can be defined as a constrained nonlinear multi-variable optimization problem. The optimization problem is defined as such:

$$\text{Maximize: } g(f_A, f_B, f_C) = \ln(\alpha) = p_A \cdot \ln(1 + f_A - f_B - f_C) +$$

$$p_B \cdot \ln(1 - f_A + f_B - f_C) + p_C \cdot \ln(1 - f_A - f_B + f_C)$$

subject to the equality constraint: $h(f_A, f_B, f_C) = f_A + f_B + f_C - 1 = 0$ with the lower bound constraints: $f_A \geq 0, f_B \geq 0, f_C \geq 0$ where $g(f_A, f_B, f_C)$ is the objective function and $h(f_A, f_B, f_C)$ is the equality constraint.

It is possible to get the optimal betting ratios, $({f_A}^* {f_B}^* {f_C}^*)$, analytically by inserting the equality constraint, $f_A = 1 - f_B - f_C$ into the objective function and converting the problem into a 2-D optimization problem.

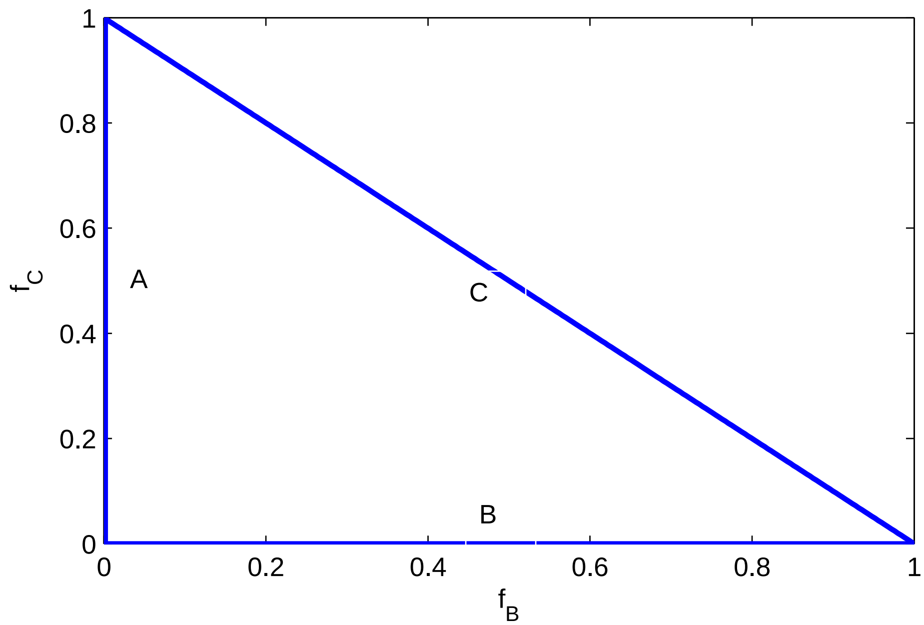
$$g(f_B, f_C) = p_A \cdot \ln(2 \cdot (1 - f_B - f_C)) + p_B \cdot \ln(2 \cdot f_B) + p_C \cdot \ln(2 \cdot f_C)$$

the domain of values of (f_B, f_C) is

$$\Delta = (f_B, f_C \in \mathbb{R}^2 : f_B + f_C \leq 1, 0 \leq f_B, 0 \leq f_C)$$

the graph of domain Δ is

Figure 15: Domain Δ



Now to find the critical point. The partial derivatives of the function g are

$$g_{f_B} = \frac{2 \cdot p_A}{2 \cdot f_B + 2 \cdot f_C - 2} + \frac{p_B}{f_B}$$

$$g_{f_C} = \frac{2 \cdot p_A}{2 \cdot f_B + 2 \cdot f_C - 2} + \frac{p_C}{f_C}$$

and the critical point is

$$f_B{}^* = p_B \qquad \text{(eq 3.3)}$$

$$f_C{}^* = p_C \qquad \text{(eq 3.4)}$$

$$f_A{}^* = p_A. \qquad \text{(eq 3.5)}$$

We can see that the optimal betting ratios are the probabilities of each horse winning the race. Next we need to prove that the optimal betting ratios $(f_B{}^*, f_C{}^*)$ are the global maximum of the function $g(f_B, f_C)$.

The domain Δ is a closed bounded domain and $g(f_B, f_C)$ is continuous in the domain except on the boundaries $(f_B = 1, f_C = 0)$ and $(f_B = 0, f_C = 1)$ because $lim_{f_B \to 1, f_C = 0}(g(f_B, f_C)) = -\infty$ and $lim_{f_C \to 1, f_B = 0}(g(f_B, f_C)) = -\infty$. Therefore the maximum of $g(f_B, f_C)$ must be in Δ.

We will define the domain that $g(f_B, f_C)$ is continuous in as the domain $\Delta^o = (f_B, f_C \in \mathbb{R}^2 : f_B + f_C < 1, 0 \leq f_B, 0 \leq f_C)$

We need to apply the second derivative test on the critical point found to know if the optimal betting ratios are relative maxima, relative minima or saddle point.

The second derivative test tells us if a critical point is a relative maxima, relative minima or saddle point. Let (x_c, y_c) be a critical point of function $f(x, y)$ and define

$$D = f_{xx}(x_c, y_c) \cdot f_{yy}(x_c, y_c) - {f_{xy}}^2(x_c, y_c)$$

We have the following cases:

• If $D > 0$ and $f_{xx}(x_c, y_c) < 0$, then $f(x, y)$ has a relative maximum at (x_c, y_c).

• If $D > 0$ and $f_{xx}(x_c, y_c) > 0$, then $f(x, y)$ has a relative minimum at (x_c, y_c).

• If $D < 0$, then $f(x, y)$ has a saddle point at (x_c, y_c).

• If $D = 0$, the second derivative test is inconclusive.

We apply the second derivative test

$$g_{f_B f_B} = -\frac{4 \cdot p_A}{(2 \cdot p_B + 2 \cdot p_C - 2)^2} - \frac{1}{p_B}$$

$$g_{f_C f_C} = -\frac{4 \cdot p_A}{(2 \cdot p_B + 2 \cdot p_C - 2)^2} - \frac{1}{p_C}$$

$$g_{f_C f_B} = g_{f_B f_C} = -\frac{4 \cdot p_A}{(2 \cdot p_B + 2 \cdot p_C - 2)^2}$$

$$D = g_{f_B f_B} \cdot g_{f_C f_C} - {g_{f_B f_C}}^2 = \frac{4 \cdot p_A}{(2 \cdot p_B + 2 \cdot p_C)^2 \cdot p_B} + \frac{4 \cdot p_A}{(2 \cdot p_B + 2 \cdot p_C)^2 \cdot p_C} + \frac{1}{p_B \cdot p_C}$$

We can see that $D > 0$ and $g_{f_B f_B} > 0$ therefore the critical point is a relative maximum. We can also prove the relative maximum is in domain Δ^o

• The first condition of the domain Δ^o is met $0 \leq {f_B}^*$

• The second condition of the domain Δ^o is met $0 \leq {f_C}^*$

- and also ${f_B}^* + {f_C}^* < 1$

therefore the relative maximum $({f_B}^*, {f_C}^*) \in \Delta^o$. The final step is to prove the relative maxima is the global maximum. First we will find the value of the function $g(f_B, f_C)$ on the the boundary of the domain Δ and at the relative maxima. The edges are labeled in figure 15. We will apply the exponential function $g(f_B, f_C)$ because $\exp(g(f_B, f_C))$ is continuous in the domain Δ.

$$\exp(g(f_B, f_C)) = \exp(p_A \cdot \ln(2 \cdot (1 - f_B - f_C)) + p_B \cdot \ln(2 \cdot f_B) + p_C \cdot \ln(2 \cdot f_C))$$

$$= \exp(p_A \cdot \ln(2 \cdot f_A)) \cdot \exp(p_B \cdot \ln(2 \cdot f_B)) \cdot \exp(p_C \cdot \ln(2 \cdot f_C))$$

$$= \exp(\ln(2 \cdot f_A)^{p_A}) \cdot \exp(\ln(2 \cdot f_B)^{p_B}) \cdot \exp(\ln(2 \cdot f_C)^{p_A})$$

$$= (2 \cdot f_A)^{p_A} \cdot (2 \cdot f_B)^{p_B} \cdot (2 \cdot f_C)^{p_C}$$

$$= 2 \cdot f_A^{p_A} \cdot f_B^{p_B} \cdot f_C^{p_C}$$

- On edge A, $f_B = 0$ and $\exp(g(f_B = 0, f_C)) = 0$.

- On edge B, $f_C = 0$ and $\exp(g(f_B, f_C = 0)) = 0$.

- On edge C, $f_C = 1 - f_B$ and after inserting this constraint into the original constraint $f_A + f_B + f_C = 1$ we get

$$f_A + f_B + 1 - f_B = 1 \rightarrow f_A = 0$$

The value of the utility function on edge C is: $\exp(g(f_B, f_C = 1 - f_B)) = 0$.

- The value of the utility function at the relative maximum is $\exp(g(f_B, f_C)) = 2 \cdot p_A^{p_A} \cdot p_B^{p_B} \cdot p_C^{p_C} > 0$

Using Weierstrass's Extreme Value Theorem, that states that a maximum of a continuous function exists in a closed bounded domain, the global maximum value of the function $g(f_B, f_C)$ is on the relative maximum point. This means that $({f_B}^*, {f_C}^*)$ are the maximum of the function $g(f_B, f_C)$.

The following figures show the level curves and the maximum point of $g(f_B, f_C)$ using different values of (p_A, p_B, p_C): (0.9, 0.05, 0.05), (0.1, 0.75, 0.15).

Figure 16: level curves of $g(f_B, f_C)$

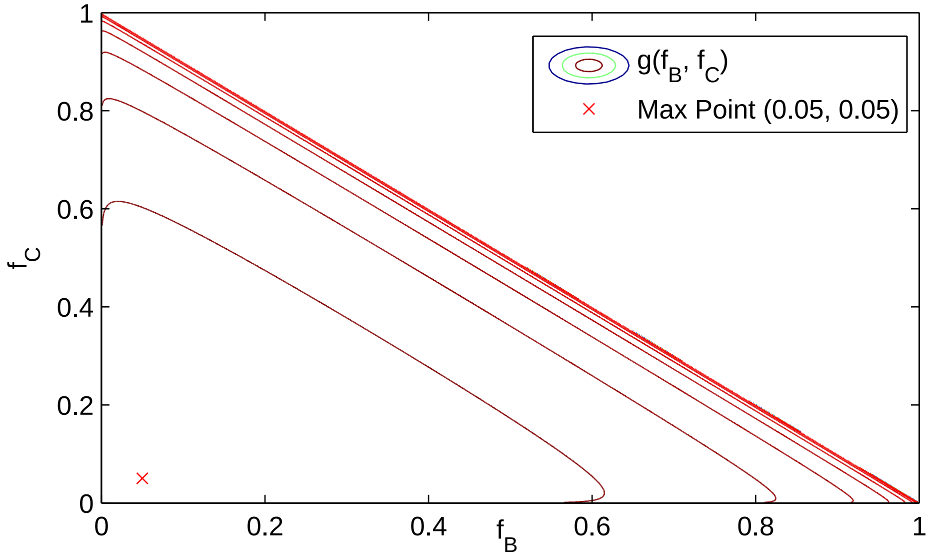
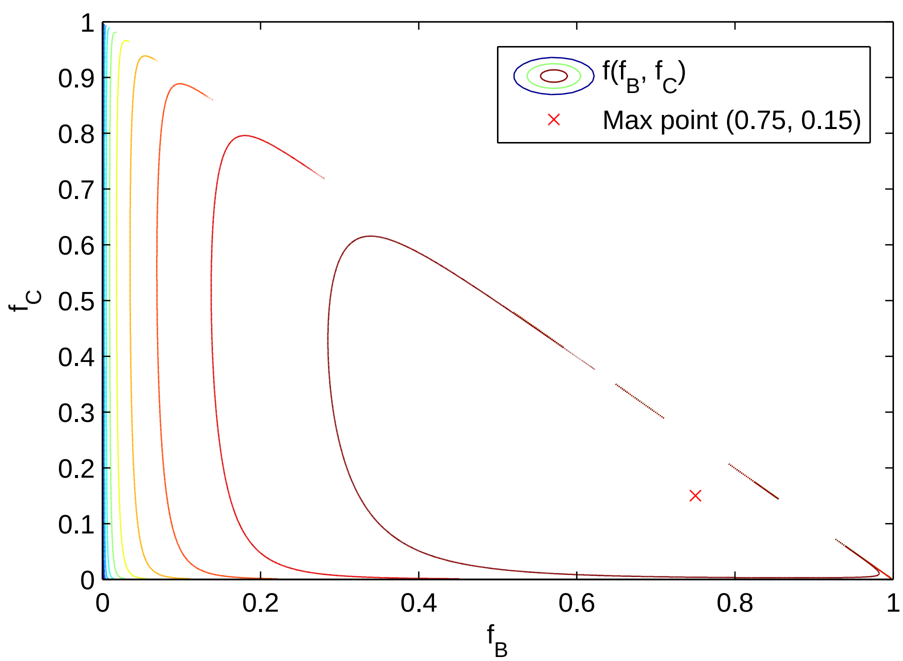


Figure 17: level curves of $g(f_B, f_C)$



In conclusion. by using (eq 3.3) - (eq 3.5) the optimal betting ratios can easily be calculated for this type of game. A final point of interest is for what values of (p_A, p_B, p_C) should the gambler play the game. Using (eq 3.2) and after inserting the optimal betting ratios ((eq 3.3) - (eq 3.5)) we get the following constraint

$$p_A \cdot \ln(2 \cdot p_A) + p_B \cdot \ln(2 \cdot p_B) + p_C \cdot \ln(2 \cdot p_C) > 0$$

We say that the gambler should play the game if he will have a positive utility function. If this constraint holds for a selected (p_A, p_B, p_C), then the gambler's geometric growth rate will be positive and he should bet in the Horse Racing game.

3.1.1 Horse Racing Simulations

This sections contains simulations that describe various scenarios of the Horse Racing game. Suppose a gambler wants to optimally bet his capital on three different horses (A, B and C) winning a horse race. The following is a list of optimal betting fractions $({f_A}^*, {f_B}^*, {f_C}^*)$ obtained using (eq 3.3) - (eq 3.5) with different values of (p_A, p_B, p_C): (0.6, 0.1, 0.3), (0.3, 0.4, 0.3), (0.1, 0.75, 0.15).

• Scenario A: $(f_A, f_B, f_C) = (0.6000, 0.1000, 0.3000)$
• Scenario B: $(f_A, f_B, f_C) = (0.30000.4000, 0.3000)$
• Scenario C: $(f_A, f_B, f_C) = (0.1000, 0.7500, 0.1500)$

In scenario A because both horse A and C have the same probability of winning then the gambler should bet the same amount on both horses and leave the rest of his capital for horse B which has the worst chance of winning. In scenarios B and C the horse that has the best chance of winning should get a bigger fraction of capital. The value of the utility function at the optimal points calculated is listed below (using (eq 3.2)) along with the geometric growth rate of capital that can be calculated using (eq 2.3).

• Scenario A: $g(0.6000, 0.1000, 0.3000) = -0.2048$ and $\alpha = 0.8148$ which is a $0.8148 - 1 = -0.1852$ or 18.52 percent negative geometric growth rate between races.
• Scenario B: $g(0.30000.4000, 0.3000) = -0.3958$ and $\alpha = 0.6732$ which is a $0.6732 - 1 = -0.3268$ or 32.68 percent negative geometric growth rate between races.
• Scenario C: $g(0.1000, 0.7500, 0.1500) = -0.0374$ and $\alpha = 0.9633$ which is a $0.9633 - 1 = -0.0367$ or 3.67 percent negative geometric growth rate between races.

We can see that the utility function values are all negative, meaning the gambler should experience negative capital growth over the long term. Figures 18, 19 and 20 each contain five simulations of each scenario above with the gambler playing $N = 100$ horse races with the data mentioned above and each figure shows the amount of capital the gambler holds after every bet.

Figure 18: Example 1

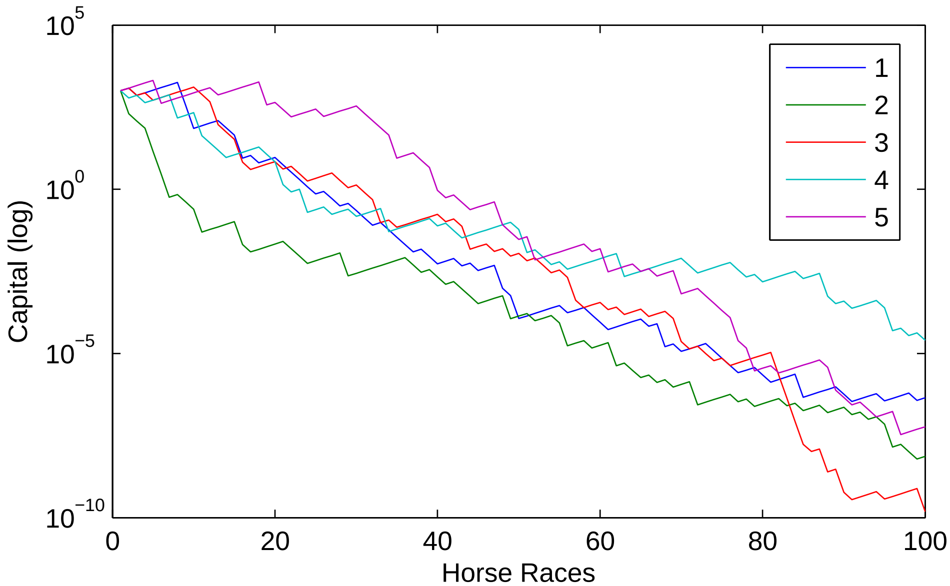


Figure 19: Example 2

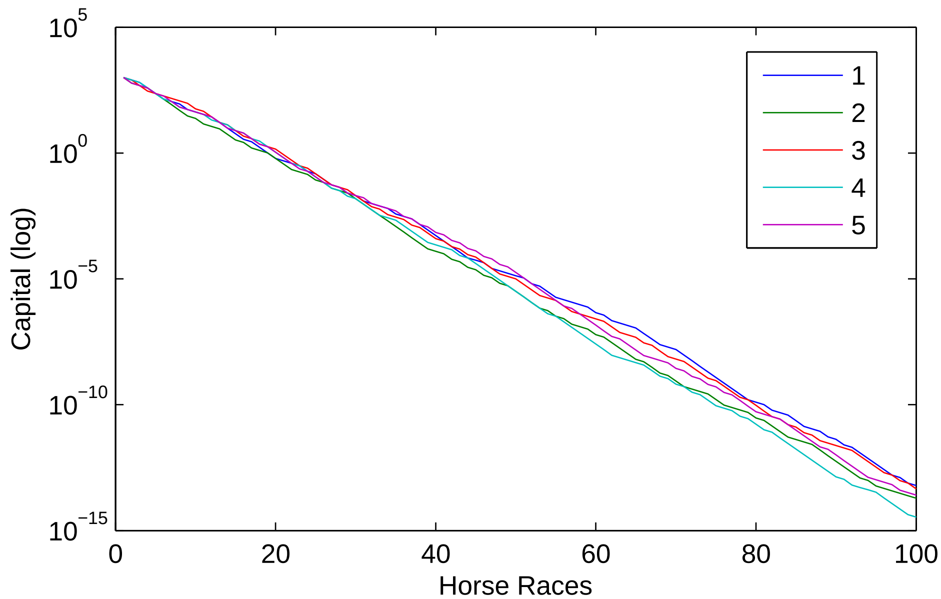
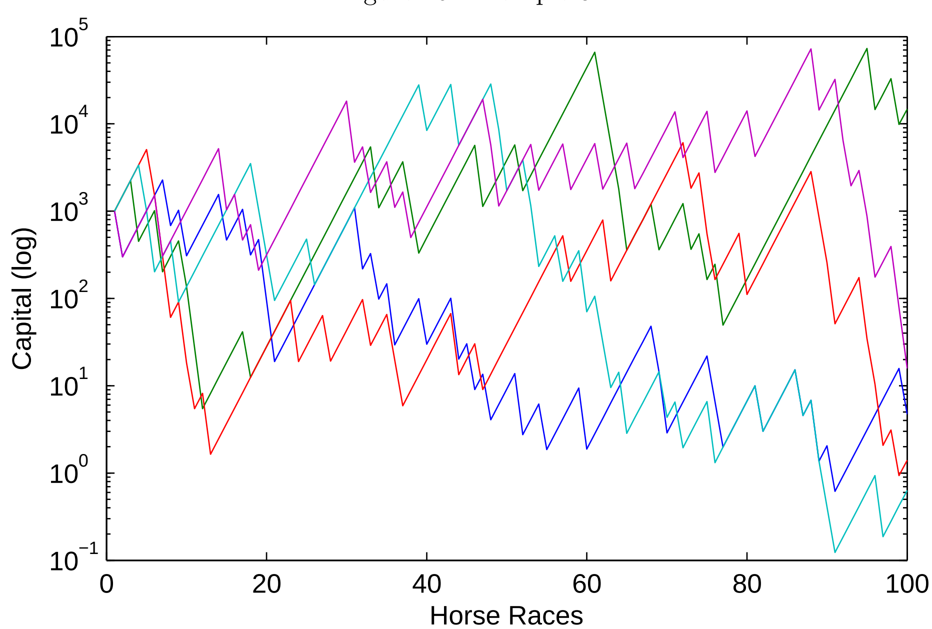


Figure 20: Example 3



It can be seen in figures 18 and 19 that the gambler's capital is in constant decline and he eventually reaches a complete loss of capital. This matches the geometric growth rate of those scenarios that was previously calculated which is below 1 percent for each scenario which means the gambler is losing capital in the long term when playing scenarios A and B mentioned above. In figure 20 there are more fluctuations in the capital because the gambler is betting 75 percent of his capital on the horse with a 75 percent chance of winning the race. The geometric growth rate calculated for scenario C is closer to 100 percent which means it is possible or the gambler to increase his capital but in the long term he will suffer more losses then gains.

Now suppose the gambler is playing three scenarios with the following probability values of (p_A, p_B, p_C): (0.9, 0.05, 0.05), (0.08, 0.04, 0.88), (0.1, 0.83, 0.07).

• Scenario D: $({f_A}^*, {f_B}^*, {f_C}^*) = (0.9000, 0.0500, 0.0500)$ and $g(0.9000, 0.0500, 0.0500) = 0.2987$ and $\alpha = 1.3482$ which is a $1.3482 - 1 = 0.3482$ or 34.82 percent positive geometric growth rate between races.
• Scenario E: $({f_A}^*, {f_B}^*, {f_C}^*) = (0.0400, 0.0800, 0.8800)$ and $g(0.0400, 0.0800, 0.8800) = 0.2498$ and $\alpha = 1.2838$ which is a $1.2838 - 1 = 0.2838$ or 28.38 percent positive geometric growth rate between races.
• Scenario F: $({f_A}^*, {f_B}^*, {f_C}^*) = (0.1000, 0.8300, 0.0700)$ and $g(0.1000, 0.8300, 0.0700) = 0.1221$ and $\alpha = 1.1299$ which is a $1.1299 - 1 = 0.1299$ or 12.99 percent positive geometric growth rate between races.

Figures 21, 22 and 23 each contain five simulations of each scenario above with the gambler playing $N = 100$ horse races with the data mentioned above and each figure shows the amount of capital the gambler holds after every bet.

Figure 21: Example 4

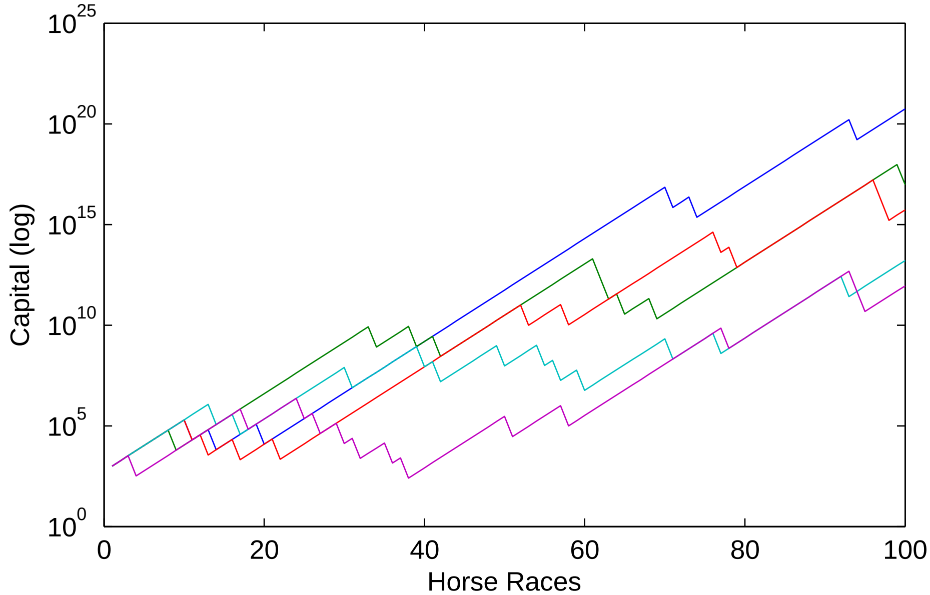


Figure 22: Example 5

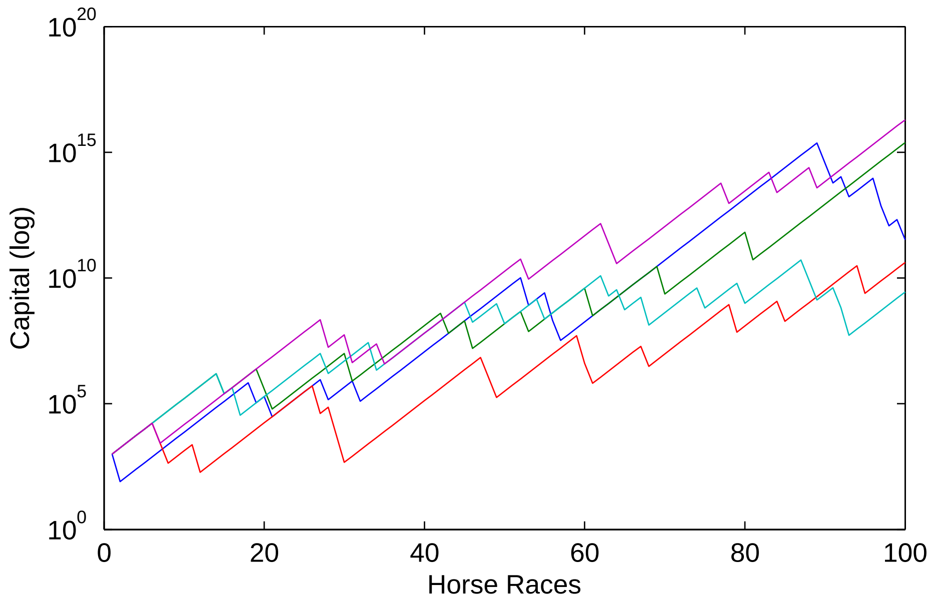
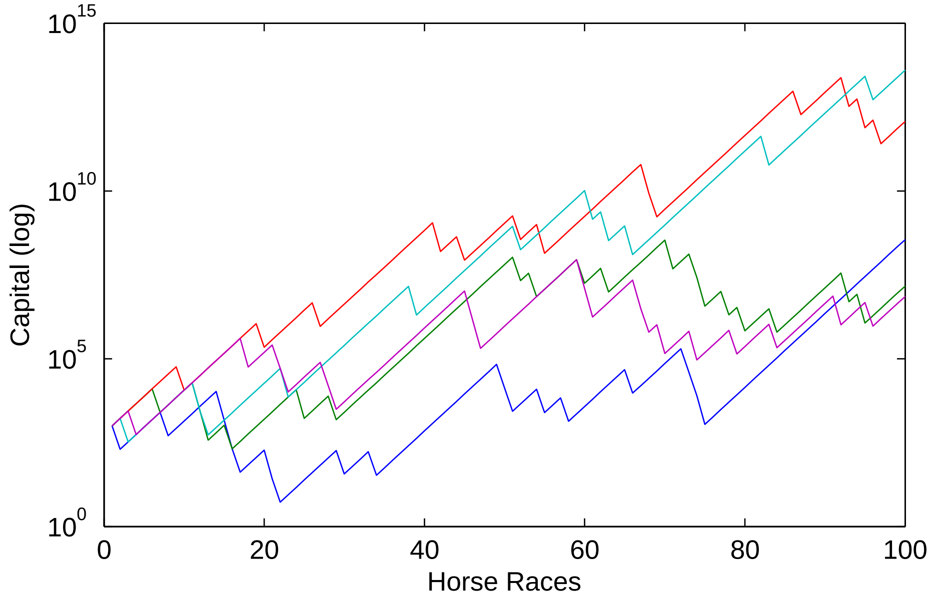


Figure 23: Example 6



In each of these current scenarios the gambler is playing a game where one of the horses has a disproportionate chance of winning the race and the other horses have very little chance of winning the race therefore the gambler bets most of his capital on that specific horse and bets the rest of his capital on the other horses. This is reflected in the optimal betting ratios calculated above. But because he is betting most of his capital on the one horse with the disproportionate chance of winning he is also exposing himself to much more risk, because if that horse loses the race then the gambler will lose most of his capital but in the long term he still increases his capital, as can be seen in the geometric growth rate calculated for each scenario above.

4 The Stock Market

The final activity discussed in this paper is the activity of investing in the stock market. In this type of activity the gambler becomes an investor that wishes to invest his capital in stock options on the stock market. He can buy or sell stock options on the market and each stock has a different probability of increasing or decreasing its value. For example, a stock can be worth 100 dollars in October and in November it can increase 10 percent to the price of 110 dollars and in December decrease by 5 percent to 104.5 dollars.

Suppose an investor has an option of buying two stocks. Each stock has a different probability of increasing in value: (p_A, p_B) and decreasing in value $(q_A = 1 - p_A, q_B = 1 - p_B)$. Both stocks can rise or fall independently of each other. Let us assume that when the stock increases in value the percentage it can increase is constant, it doesn't change between months, and at the end of the month the investor has a capital increase of what he invested into buying the stock multiplied by a factor of how much the stock's value increased. For example, using the previous example, the investor would have 110 dollars in November because the stock increased ten percent. If the stock increases again the next month (by 10 percent because the percentage it increases is constant) then he would have 121 dollars. The same applies if the stock lowers in price. If the stock rose in value then the investor's capital is the amount of capital he invested in the stock multiplied by $V \in (1, \infty)$ and if the stock fell in value then the decrease in capital for the investor is the amount of capital he invested multiplied by the fraction $Z \in (0, 1)$. This case strongly resembles the case of the general coin toss but in this case it is defined as a coin toss game of V to Z odds for each stock option. The fraction of capital he invests on each stock is $f_A, f_B \in [0, 1], f_A + f_B = 1$.

For example, if the investor were to invest in the two stocks and the next month the value of stock A rose and the value of stock B fell then his current capital would be:

$$X_1 = X_0 \cdot (V_A \cdot f_A + Z_B \cdot f_B)$$

If in the next month stock B rose and stock A fell

$$X_2 = X_1 \cdot (Z_A \cdot f_A + V_B \cdot f_B) = X_0 \cdot (V_A \cdot f_A + Z_B \cdot f_B) \cdot (Z_A \cdot f_A + V_B \cdot f_B).$$

The investor is investing in stock options for a series of $N \in \mathbb{N}$ months with

• $WL \in \mathbb{N}$ - number of months in which stock A rose in value and B fell in value.
• $LW \in \mathbb{N}$ - number of months in which stock B rose in value and A fell in value.
• $WW \in \mathbb{N}$ - number of months in which stocks A and B rose in value together.
• $LL \in \mathbb{N}$ - number of months in which stocks A and B fell in value together.
with $WL + LW + WW + LL = N$. The equation describing the current capital of the investor is

$$X_N = X_0 \cdot (V_A \cdot f_A + Z_B \cdot f_B)^{WL} \cdot (Z_A \cdot f_A + V_B \cdot f_B)^{LW} \cdot (V_A \cdot f_A + V_B \cdot f_B)^{WW} \cdot (Z_A \cdot f_A + Z_B \cdot f_B)^{LL}. \qquad \text{(eq 4.1)}$$

The next step is defining the utility function

$$G(f_A, f_B) = \ln(\tfrac{X_N}{X_0}) = WL \cdot \ln(V_A \cdot f_A + Z_B \cdot f_B) + LW \cdot \ln(Z_A \cdot f_A + V_B \cdot f_B) + WW \cdot \ln(V_A \cdot f_A + V_B \cdot f_B) + LL \cdot \ln(Z_A \cdot f_A + Z_B \cdot f_B)$$

as before this is the logarithmic capital growth function of N discrete series of investments. We divide by N

$$g(f_A, f_B) = \frac{1}{N} \cdot G(f_A, f_B) = \frac{1}{N} \cdot \ln(\frac{X_N}{X_0}) = \frac{WL}{N} \cdot \ln(V_A \cdot f_A + Z_B \cdot f_B) +$$

$$\frac{LW}{N} \cdot \ln(Z_A \cdot f_A + V_B \cdot f_B) + \frac{WW}{N} \cdot \ln(V_A \cdot f_A + V_B \cdot f_B) +$$

$$\frac{LL}{N} \cdot \ln(Z_A \cdot f_A + Z_B \cdot f_B)$$

and after a very long series of investments

$$g(f_A, f_B) = lim_{N\to\infty}(\frac{1}{N} \cdot \ln(\frac{X_N}{X_0})) = lim_{N\to\infty}(\frac{WL}{N}) \cdot \ln(V_A \cdot f_A + Z_B \cdot f_B) +$$

$$lim_{N\to\infty}(\frac{LW}{N}) \cdot \ln(Z_A \cdot f_A + V_B \cdot f_B) + lim_{N\to\infty}(\frac{WW}{N}) \cdot \ln(V_A \cdot f_A + V_B \cdot f_B) +$$

$$lim_{N\to\infty}(\frac{LL}{N}) \cdot \ln(Z_A \cdot f_A + Z_B \cdot f_B)$$

When we divide WL by the number of months overall the investor has spent investing in stocks A and B (assuming N is a very large number) and using the Law of Large Numbers we get $lim_{N\to\infty}(\frac{WL}{N}) = p_A \cdot q_B$. The same goes for LW, WW, LL.

$$\begin{gathered} g(f_A, f_B) = p_A \cdot q_B \cdot \ln(V_A \cdot f_A + Z_B \cdot f_B) + \\ q_A \cdot p_B \cdot \ln(Z_A \cdot f_A + V_B \cdot f_B) + p_A \cdot p_B \cdot \ln(V_A \cdot f_A + V_B \cdot f_B) + \\ q_A \cdot q_B \cdot \ln(Z_A \cdot f_A + Z_B \cdot f_B) \end{gathered} \qquad \text{(eq 4.2)}$$

The next step is to find the maximum of the utility function to get the optimal betting ratios. This problem can be defined as a nonlinear constrained optimization problem and it can be solved in MATLAB. The optimization problem is defined

Maximize:

$$g(f_A, f_B) = p_A \cdot q_B \cdot \ln(V_A \cdot f_A + Z_B \cdot f_B) + q_A \cdot p_B \cdot \ln(Z_A \cdot f_A + V_B \cdot f_B) +$$

$$p_A \cdot p_B \cdot \ln(V_A \cdot f_A + V_B \cdot f_B) + q_A \cdot q_B \cdot \ln(Z_A \cdot f_A + Z_B \cdot f_B)$$

subject to the equality constraint: $h(f_A, f_B) = f_A + f_B - 1 = 0$ with the upper bound constraints: $f_A, f_B \geq 0$ where $g(f_A, f_B)$ is the objective function and $h(f_A, f_B)$ is the equality constraint. It is also possible to insert the constraint $h(f_A, f_B)$ into the objective equation and attempt to solve it analytically.

$$g(f_A) = p_A \cdot p_B \cdot \ln(V_A \cdot f_A + V_B \cdot (1 - f_A)) + p_A \cdot q_B \cdot \ln(V_A \cdot f_A + Z_B \cdot (1 - f_A)) +$$

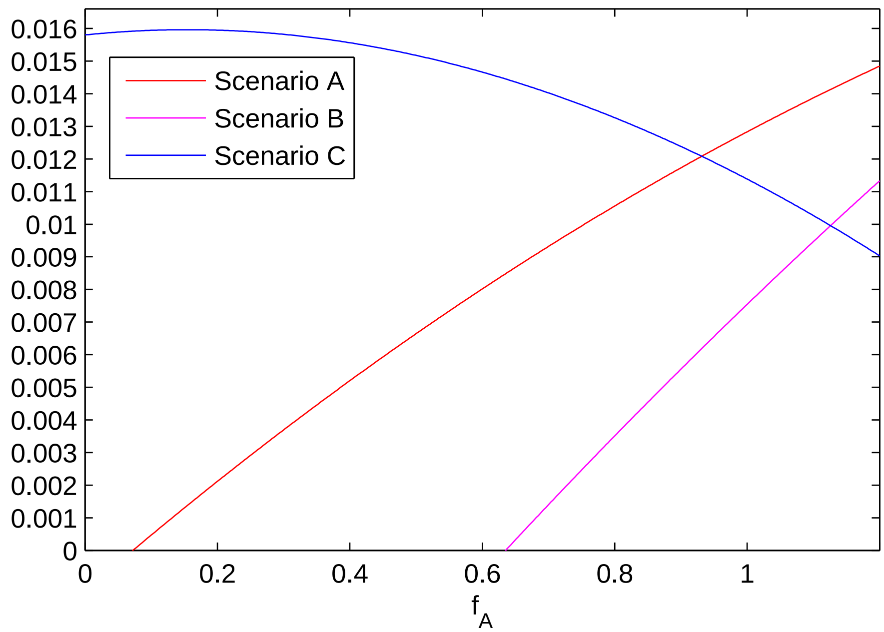
$$p_B \cdot q_A \cdot \ln(Z_A \cdot f_A + V_B \cdot (1 - f_A)) + q_A \cdot q_B \cdot \ln(Z_A \cdot f_A + Z_B \cdot (1 - f_A))$$

the domain of values of (f_A) is

$$\Delta = (f_A \in \Delta : 0 \leq f_A \leq 1)$$

The following figure is a graph of $g(f_A)$ using the parameters listed in the Stock Market Examples section below.

Figure 24: Graph of $g(f_A)$



Now to find the critical point. The first derivative of the function g is

$$g'(f_A) = \frac{p_A \cdot p_B \cdot (V_A - V_B)}{V_A \cdot f_A + V_B \cdot (1 - f_A)} + \frac{p_A \cdot q_B \cdot (V_A - Z_B)}{V_A \cdot f_A + Z_B \cdot (1 - f_A)}$$

$$- \frac{p_B \cdot q_A \cdot (V_B - Z_A)}{Z_A \cdot f_A + V_B \cdot (1 - f_A)} + \frac{q_A \cdot q_B \cdot (Z_A - Z_B)}{Z_A \cdot f_A + Z_B \cdot (1 - f_A)} = 0$$

This is a difficult equation to solve therefore we will not continue attempting to solve this problem analytically, we will use the MATLAB function *fmincon* which is used for finding a constrained minimum of a function of several variables.

4.1 Stock Market Simulations

This section contains simulations that describe various scenarios of the Stock Market game. Suppose an investor wants to invest in two stocks (A and B). Each stock has a probability of rising in value at the end of every month. Stock A has a p_A probability of rising in value and stock B has a p_B probability of rising in value. If stock A increases in value it increases by $(V_A - 1) \cdot 100$ percent and if it falls it falls by $(1 - Z_A) \cdot 100$ percent. If stock B increases in value it increases by $(V_B - 1) \cdot 100$ percent and if it falls it falls by $(1 - Z_B) \cdot 100$ percent. The following is a list of scenarios using different values of $(p_A, p_B, V_A, Z_A, V_B, Z_B)$. The graphs of the utility function of each scenario are in figure 24.

• Scenario 1 - Stock A: $p_A = 0.5, q_A = 0.5, V_A = 1.08, Z_A = 0.95$, Stock B: $p_B = 0.5, q_B = 0.5, V_B = 1.05, Z_B = 0.95$
• Scenario 2 - Stock A: $p_A = 0.5, q_A = 0.5, V_A = 1.08, Z_A = 0.94$, Stock B: $p_B = 0.65, q_B = 0.35, V_B = 1.01, Z_B = 0.94$
• Scenario 3 - Stock A: $p_A = 0.3, q_A = 0.7, V_A = 1.20, Z_A = 0.94$, Stock B: $p_B = 0.9, q_B = 0.1, V_B = 1.02, Z_B = 0.98$

Using the *fmincon* function in MATLAB for solving constrained nonlinear optimization problems the following optimal betting fractions and geometric growth rates were calculated.

• Scenario 1 - ${f_A}^* = 1, {f_B}^* = 0$ and $g(1, 0) = 0.0128$ and $\alpha = 1.0129$ which is a geometric growth rate of 1.29 percent.
• Scenario 2 - ${f_A}^* = 1, {f_B}^* = 0$ and $g(1, 0) = 0.0075$ and $\alpha = 1.0075$ which is a geometric growth rate of 0.76 percent.
• Scenario 3 - ${f_A}^* = 0.1492, {f_B}^* = 0.8508$ and $g(0.1492, 0.8508) = 0.0160$ and $\alpha = 1.0161$ which is a geometric growth rate of 1.61 percent.

Figures 24, 25 and 26 each contain five simulations of each scenario (1 ,2, 3) over $N = 100$ months with the data mentioned above and shows the amount of capital the investor holds after every investment. The first figure contains five simulations of scenario 1, the second of scenario 2 and the third of scenario 3

Figure 25: Example 1

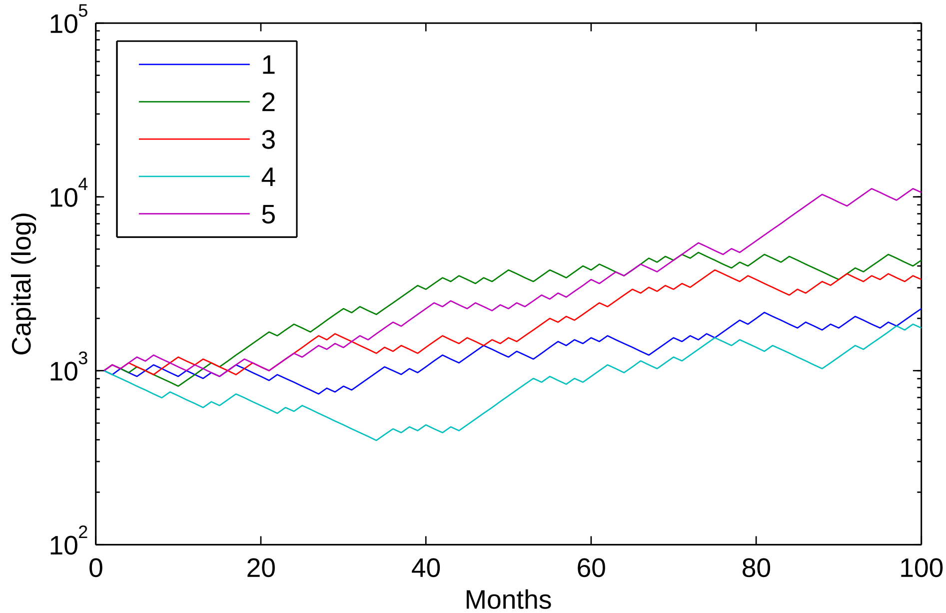


Figure 26: Example 2

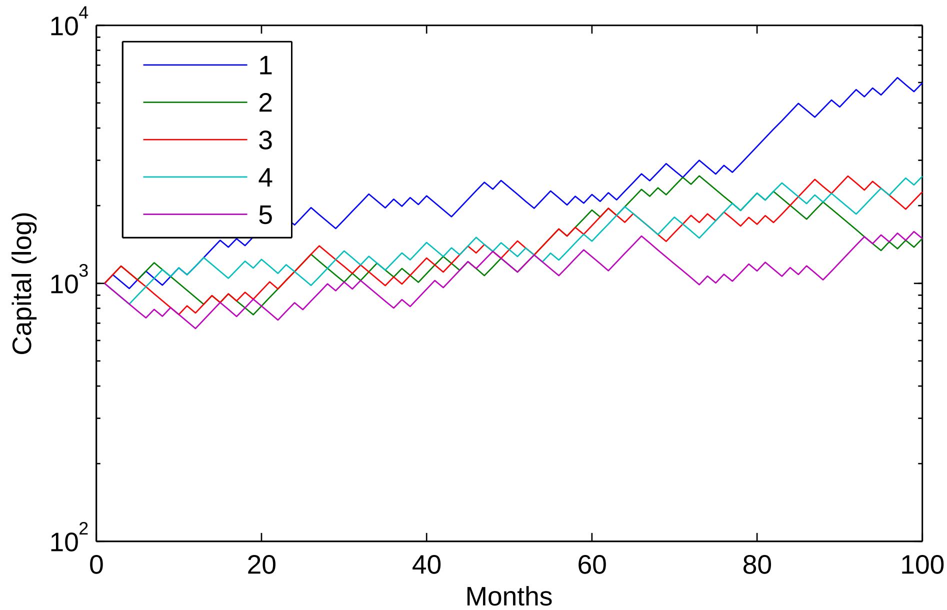
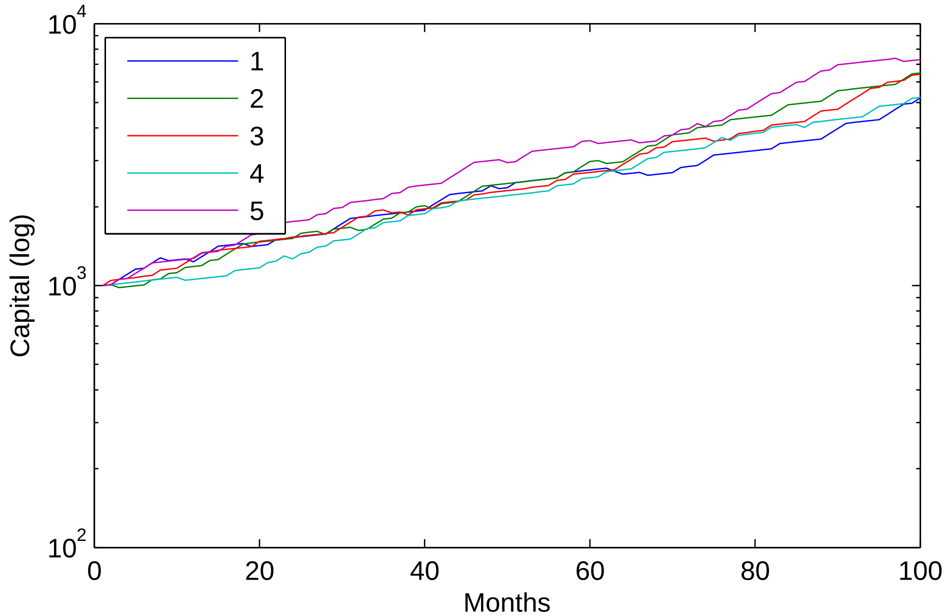


Figure 27: Example 3



The geometric growth rate of scenario 1 calculated above is 1.29 percent and if we look at figure 25 we can see the investors capital grow at a small rate in all the simulations. The optimal betting ratios calculated are ${f_A}^* = 1, {f_B}^* = 0$ meaning the optimal point of the function is on the edge of the interval $[0, 1]$ meaning the investor should invest only in one stock (stock A). In figure 26 we can see the investors capital grow at a smaller rate then in the previous figure, which matches the smaller growth rate calculated for scenario 2 and the optimal betting ratios calculated are the same as in scenario 1. In figure 27 we get the best growth rate of all the scenarios. We can clearly see that the optimal betting ratios and the geometric growth rates calculated for each scenario are compatible to what we see in figure 24.

What can be derived from figures 25 and 27 about the probabilities of the stock options is that, the higher the probability of a stock option to rise, the more Kelly tells the investor to bet only on that option, if the probability is high enough.

Another point of interest is how the parameters V, Z effect the optimal betting ratios. In figure 26, stock B has a better probability of rising in price then stock A, but stock A has a larger increase percentage then stock B. This tells us that even if the probability of stock X rising is lower then stock Y rising, but the increase percentage of stock X is much higher, then Kelly tells us to bet more on stock option X.

5 Concluding Remarks

The purpose of this paper was to present an approach of how to find an optimal amount of capital to gamble in a bet. We examined the approach of maximizing the expected value of a gamblers capital and we saw that that approach could easily lead the gambler to financial ruin. We next analyzed the Kelly Criterion approach and proved that it lowers the risk of financial ruin to the gambler to zero and that it is possible to easily calculate optimal betting ratios for a game of Coin Toss. We later expanded to the case of betting in games of Horse Racing and to investing in the Stock Market and found that it is possible to calculate optimal betting ratios for these types of games although in the case of the Stock Market the computations get substantially more complex the more stocks we wish to bet on simultaneously, in each game respectively. It also possible to apply the Kelly Criterion to other types of games.

We created mathematical models for the Coin Toss, Horse Racing and Stock Market games and we used MATLAB to simulate different scenarios of those games and we saw that the Kelly Criterion is effective but mostly for long term games. For short term games the increase in capital is quite small and there is risk in ending a series of short term bets with a final amount of capital being less then the initial amount of capital. This makes this approach less feasible and hard to apply to many real life gambling situations. In most cases when playing long term games, where a great deal of time has passed, the Kelly approach generates a higher rate of return.

6 Bibliography

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[2] Ravi Phataford, Some aspects of gambling with the Kelly Criterion, *Monash University*, http://www.cmss.monash.edu.au/assets/files/kelly150610.pdf.

[3] Louis M. Rotando and Edward O, Thorp, The Kelly Criterion and the Stock Market, *The American Mathematical Monthly*, Volume 99, Number 10, December 1992.

[4] Edward O. Thorp, The Kelly Criterion in Blackjack, Sports Betting and the Stock Market, *The 10th International Conference On Gambling and Risk Taking*, Montreal, June 1997.

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7 Addendum

Matlab code of Coin Toss examples

```
clear
x_0 = 1000;
num_of_simulations = 5;
N = 100;
p = 0.65;
f = 0.3;

for sim_num = 1:num_of_simulations

    rand_vector = rand(N,1);
    for i = 1:N
        if rand_vector(i) < p
            rand_arrays(sim_num, i) = 1;
        else
            rand_arrays(sim_num, i) = 0;
        end
    end

    capital_arrays(1, sim_num) = x_0;
    for i = 2:N
        if rand_arrays(sim_num, i) == 1
            capital_arrays(i, sim_num) = capital_arrays(i - 1, sim_num) * (1 + f);
        elseif rand_arrays(sim_num, i) == 0
            capital_arrays(i, sim_num) = capital_arrays(i - 1, sim_num) * (1 - f);
        end
    end
end

plot(capital_arrays);
```

Matlab code of Horse Racing examples

```
clear
x_0 = 1000;
N = 100;
num_of_simulations = 5;

pA_1 = 0.1;
pA_2 = 0.83;
pA_3 = 0.07;
f1_A = pA_1;
f2_A = pA_2;
f3_A = pA_3;

for sim_num = 1:num_of_simulations

    rand_vector = rand(N,1);
    for i=1:N
        if rand_vector(i) >= 0 && rand_vector(i) <= pA_1
            rand_arrays(sim_num, i) = 1;
        end
        if rand_vector(i) > pA_1 && rand_vector(i) <= (pA_1 + pA_2)
```

```
            rand_arrays(sim_num, i) = 2;
        end
        if rand_vector(i) > (pA_1 + pA_2) && rand_vector(i) <=1
            rand_arrays(sim_num, i) = 3;
        end
    end

    capital_arrays(1, sim_num) = x_0;
    for i = 2:N
        if rand_arrays(sim_num, i) == 1
            capital_arrays(i, sim_num) = capital_arrays(i - 1, sim_num) * (1 + f1_A - f2_A - f3_A);
        elseif rand_arrays(sim_num, i) == 2
            capital_arrays(i, sim_num) = capital_arrays(i - 1, sim_num) * (1 - f1_A + f2_A - f3_A);
        elseif rand_arrays(sim_num, i) == 3
            capital_arrays(i, sim_num) = capital_arrays(i - 1, sim_num) * (1 - f1_A - f2_A + f3_A);
        end
    end
end

% plot
plot(capital_arrays);
```

Matlab code of Stock Market examples

```
clear
x_0 = 1000;
N = 100;
num_of_simulations = 5;

% 1 = A rises B falls
% 2 = A falls B rises
% 3 = A rises B rises
% 4 = A falls B falls

% scenario A
p_A = 0.5;
q_A = 0.5;
p_B = 0.5;
q_B = 0.5;
V_A = 1.08;
Z_A = 0.95;
V_B = 1.05;
Z_B = 0.95;

fA = 1;
fB = 0;

for sim_num = 1:num_of_simulations

    rand_vector_A = rand(N,1);
    for i = 1:N
        if rand_vector_A(i) < p_A
            rand_arrays_A(sim_num, i) = 1;
        else
            rand_arrays_A(sim_num, i) = 0;
        end
    end
```

```
    rand_vector_B = rand(N,1);
    for i = 1:N
        if rand_vector_B(i) < p_B
            rand_arrays_B(sim_num, i) = 1;
        else
            rand_arrays_B(sim_num, i) = 0;
        end
    end

    % compute scenario
    capital_arrays(1, sim_num) = x_0;

    for i = 2:N
        if rand_arrays_A(sim_num, i) == 1 && rand_arrays_B(sim_num, i) == 0
            capital_arrays(i, sim_num) = capital_arrays(i - 1, sim_num) * (V_A * fA + Z_B * fB);
        elseif rand_arrays_A(sim_num, i) == 0 && rand_arrays_B(sim_num, i) == 1
            capital_arrays(i, sim_num) = capital_arrays(i - 1, sim_num) * (Z_A * fA + V_B * fB);
        elseif rand_arrays_A(sim_num, i) == 1 && rand_arrays_B(sim_num, i) == 1
            capital_arrays(i, sim_num) = capital_arrays(i - 1, sim_num) * (V_A * fA + V_B * fB);
        elseif rand_arrays_A(sim_num, i) == 0 && rand_arrays_B(sim_num, i) == 0
            capital_arrays(i, sim_num) = capital_arrays(i - 1, sim_num) * (Z_A * fA + Z_B * fB);
        end
    end
end

% plot
plot(capital_arrays);
```